EJERCICIOS DE INTEGRALES INDEFINIDAS

Ejercicio 1.-

Calcula las siguientes integrales:

a) $\int 7x^4$ b) $\int \frac{1}{x^2}$ c) $\int \sqrt{x}$

d) $\int \sqrt[3]{5x^2}$ e) $\int \frac{\sqrt[3]{x} + \sqrt{5x^3}}{3x}$ f) $\int \frac{\sqrt{5x^3}}{\sqrt[3]{3x}}$

a) $\int 7x^4 = 7\,\frac{x^5}{5} + k = \frac{7x^5}{5} + k$

b) $\int \frac{1}{x^2} = \int x^{-2} = \frac{x^{-1}}{-1} + k = \frac{-1}{x} + k$

c) $\int \sqrt{x} = \int x^{1/2} = \frac{x^{3/2}}{3/2} + k = \frac{2\sqrt{x^3}}{3} + k$

d) $\int \sqrt[3]{5x^2} = \int \sqrt[3]{5}\, x^{2/3} = \sqrt[3]{5}\,\frac{x^{5/3}}{5/3} + k = \frac{3\sqrt[3]{5x^5}}{5} + k$

e) $\int \frac{\sqrt[3]{x} + \sqrt{5x^3}}{3x} = \int \frac{x^{1/3}}{3x} + \int \frac{\sqrt{5}x^{3/2}}{3x} = \frac{1}{3}\int x^{-2/3} + \frac{\sqrt{5}}{3}\int x^{1/2} =$

$= \frac{1}{3}\,\frac{x^{1/3}}{1/3} + \frac{\sqrt{5}}{3}\,\frac{x^{3/2}}{3/2} + k = \sqrt[3]{x} + \frac{2\sqrt{5x^3}}{9} + k$

f) $\int \frac{\sqrt{5x^3}}{\sqrt[3]{3x}} = \int \frac{\sqrt{5}\cdot x^{3/2}}{\sqrt[3]{3}\cdot x^{1/3}} = \frac{\sqrt{5}}{\sqrt[3]{3}}\int x^{7/6} = \frac{\sqrt{5}}{\sqrt[3]{3}}\,\frac{x^{13/6}}{13/6} + k = \frac{6\sqrt{5}\,\sqrt[6]{x^{13}}}{13\sqrt[3]{3}} + k$

Ejercicio 2.-

Calcula:

a) $\int \frac{x^4 - 5x^2 + 3x - 4}{x}$ b) $\int \frac{x^4 - 5x^2 + 3x - 4}{x + 1}$

c) $\int \frac{x^4 - 5x^2 + 3x - 4}{x^2 + 1}$ d) $\int \frac{x^3}{x - 2}$

a) $\int \frac{x^4 - 5x^2 + 3x - 4}{x} = \int \left(x^3 - 5x + 3 - \frac{4}{x}\right) = \frac{x^4}{4} - \frac{5x^2}{2} + 3x - 4\ ln\ |x| + k$

b) $\int \frac{x^4 - 5x^2 + 3x - 4}{x + 1} = \int \left(x^3 - x^2 - 4x + 7 - \frac{11}{x + 1}\right) =$

$$= \frac{x^4}{4} - \frac{x^3}{3} - 2x^2 + 7x - 11\ ln\ |x + 1| + k$$

c) $\int \frac{x^4 - 5x^2 + 3x - 4}{x^2 + 1} = \int \left(x^2 - 6 + \frac{3x + 2}{x^2 + 1}\right) = \int \left(x^2 - 6 + \frac{3x}{x^2 + 1} + \frac{2}{x^2 + 1}\right) =$

$$= \int x^2 - \int 6 + \frac{3}{2}\int \frac{2x}{x^2 + 1} + 2\int \frac{1}{x^2 + 1} =$$

$$= \frac{x^3}{3} - 6x + \frac{3}{2}\ ln\ (x^2 + 1) + 2\ arc\ tg\ x + k$$

d) $\int \frac{x^3}{x - 2} = \int \left(x^2 + 2x + 4 + \frac{8}{x - 2}\right) = \frac{x^3}{3} + x^2 + 4x + 8\ ln\ |x - 2| + k$

Ejercicio 3.-

Calcula:

a) $\int \cos^4 x\ sen\ x\ dx$ **b)** $\int 2^{sen\ x} \cos x\ dx$

a) $\int \cos^4 x\ sen\ x\ dx = -\int \cos^4 x\,(-sen\ x)\ dx = -\frac{\cos^5 x}{5} + k$

b) $\int 2^{sen\ x} \cos x\ dx = \frac{1}{ln\ 2}\int 2^{sen\ x} \cos x \cdot ln\ 2\ dx = \frac{2^{sen\ x}}{ln\ 2} + k$

Ejercicio 4.-

Calcula:

a) $\int cotg\ x\ dx$ **b)** $\int \frac{5x}{x^4 + 1} dx$

a) $\int cotg\ x\ dx = \int \frac{\cos x}{sen\ x}\ dx = ln\ |sen\ x| + k$

b) $\int \frac{5x}{x^4 + 1}\ dx = \frac{5}{2}\int \frac{2x}{1 + (x^2)^2}\ dx = \frac{5}{2}\ arc\ tg\ (x^2) + k$

Ejercicio 5.-

Calcula: $\int \frac{1}{\sqrt[3]{x^2} - \sqrt{x}}\, dx$

Hacemos el cambio $x = t^6$, $dx = 6t^5\, dt$:

$$\int \frac{1}{\sqrt[3]{x^2} - \sqrt{x}}\, dx = \int \frac{1}{\sqrt[3]{t^{12}} - \sqrt{t^6}}\, 6t^5\, dt = \int \frac{6t^5}{t^4 - t^3}\, dt = \int \frac{6t^2}{t-1}\, dt = 6\int \frac{t^2}{t-1}\, dt =$$

$$= 6\int\left(t + 1 + \frac{1}{t-1}\right) dt = 6\int\left(\frac{t^2}{2} + t - ln\,|t-1|\right) + k =$$

$$= 6\left(\frac{\sqrt[6]{x^2}}{2} + \sqrt[6]{x} - ln\,|\sqrt[6]{x} - 1|\right) + k = 3\sqrt[3]{x} + 6\sqrt[6]{x} - 6\, ln\,|\sqrt[6]{x} - 1| + k$$

Ejercicio 6.-

Calcula: $\int \frac{x}{\sqrt{1-x^2}}\, dx$

Hacemos el cambio $\sqrt{1-x^2} = t \;\rightarrow\; 1 - x^2 = t^2 \;\rightarrow\; x = \sqrt{1-t^2}$

$$dx = \frac{-t}{\sqrt{1-t^2}}\, dt$$

$$\int \frac{x}{\sqrt{1-x^2}}\, dx = \int \frac{\sqrt{1-t^2}}{t^2} \cdot \frac{-t}{\sqrt{1-t^2}}\, dt = \int -1\, dt = -t + k = -\sqrt{1-x^2} + k$$

Ejercicio 7.-

Calcula: $\int x\, sen\, x\, dx$

Llamamos $I = \int x\, sen\, x\, dx$.

$$\left.\begin{array}{l} u = x, \quad du = dx \\ dv = sen\, x\, dx, \quad v = -cos\, x \end{array}\right\} \; I = -x\, cos\, x + \int cos\, x\, dx = -x\, cos\, x + sen\, x + k$$

Ejercicio 8.-

Calcula: $\int x\, arc\, tg\, x\, dx$

Llamamos $I = \int x\, arc\, tg\, x\, dx$.

$$\left.\begin{array}{l} u = arc\, tg\, x, \quad du = \frac{1}{1+x^2}\, dx \\ dv = x\, dx, \quad v = \frac{x^2}{2} \end{array}\right\}$$

$$I = \frac{x^2}{2}\, arc\, tg\, x - \frac{1}{2}\int\left(\frac{x^2}{1+x^2}\right) dx = \frac{x^2}{2}\, arc\, tg\, x - \frac{1}{2}\int\left(1 - \frac{1}{1+x^2}\right) dx =$$

$$= \frac{x^2}{2}\, arc\, tg\, x - \frac{1}{2}[x - arc\, tg\, x] + k = \frac{x^2}{2}\, arc\, tg\, x - \frac{1}{2}x + \frac{1}{2}\, arc\, tg\, x + k =$$

$$= \frac{x^2+1}{2}\, arc\, tg\, x - \frac{1}{2}x + k$$

Ejercicio 9.-

Calcula: $\int \frac{3x^2 - 5x + 1}{x - 4}\, dx$

$$\int \frac{3x^2 - 5x + 1}{x - 4}\, dx = \int \left(3x + 7 + \frac{29}{x - 4}\right) dx = \frac{3x^2}{2} + 7x + 29\, ln\, |x - 4| + k$$

Ejercicio 10.-

Calcula:

a) $\int \frac{5x - 3}{x^3 - x}\, dx$ **b)** $\int \frac{x^2 - 2x + 6}{(x - 1)^3}\, dx$

a) Descomponemos la fracción:

$$\frac{5x - 3}{x^3 - x} = \frac{5x - 3}{x(x - 1)(x + 1)} = \frac{A}{x} + \frac{B}{x - 1} + \frac{C}{x + 1}$$

$$\frac{5x - 3}{x^3 - x} = \frac{A(x - 1)(x + 1) + Bx(x + 1) + Cx(x - 1)}{x(x - 1)(x + 1)}$$

$$5x - 3 = A(x - 1)(x + 1) + Bx(x + 1) + Cx(x - 1)$$

Hallamos A, B y C dando a x los valores 0, 1 y –1:

$$\left.\begin{array}{lllll} x = 0 & \Rightarrow & -3 = -A & \Rightarrow & A = 3 \\ x = 1 & \Rightarrow & 2 = 2B & \Rightarrow & B = 1 \\ x = -1 & \Rightarrow & -8 = 2C & \Rightarrow & C = -4 \end{array}\right\}$$

Así, tenemos que:

$$\int \frac{5x - 3}{x^3 - x}\, dx = \int \left(\frac{3}{x} + \frac{1}{x - 1} - \frac{4}{x + 1}\right) dx = 3\, ln|x| + ln|x - 1| - 4\, ln|x + 1| + k$$

b) Descomponemos la fracción:

$$\frac{x^2 - 2x + 6}{(x - 1)^3} = \frac{A}{x - 1} + \frac{B}{(x - 1)^2} + \frac{C}{(x - 1)^3} = \frac{A(x - 1)^2 + B(x - 1) + C}{(x - 1)^3}$$

$$x^2 - 2x + 6 = A(x - 1)^2 + B(x - 1) + C$$

Dando a x los valores 1, 0 y 2, queda:

$$\left.\begin{array}{lll} x = 1 & \Rightarrow & 5 = C \\ x = 0 & \Rightarrow & 6 = A - B + C \\ x = 2 & \Rightarrow & 6 = A + B + C \end{array}\right\} \begin{array}{l} A = 1 \\ B = 0 \\ C = 5 \end{array}$$

Por tanto:

$$\int \frac{x^2 - 2x + 6}{(x - 1)^3}\, dx = \int \left(\frac{1}{x - 1} + \frac{5}{(x - 1)^3}\right) dx = ln|x - 1| - \frac{5}{2(x - 1)^2} + k$$

Ejercicio 11.-

Calcula:

a) $\displaystyle\int \frac{x^3 + 22x^2 - 12x + 8}{x^4 - 4x^2}\, dx$ **b)** $\displaystyle\int \frac{x^3 - 4x^2 + 4x}{x^4 - 2x^3 - 4x^2 + 8x}\, dx$

a) $x^4 - 4x^2 = x^2(x^2 - 4) = x^2(x - 2)(x + 2)$

Descomponemos la fracción:

$$\frac{x^3 + 22x^2 - 12x + 8}{x^2(x - 2)(x + 2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x - 2} + \frac{D}{x + 2}$$

$$\frac{x^3 + 22x^2 - 12x + 8}{x^2(x - 2)(x + 2)} =$$

$$= \frac{Ax(x - 2)(x + 2) + B(x - 2)(x + 2) + Cx^2(x + 2) + Dx^2(x - 2)}{x^2(x - 2)(x + 2)}$$

$$x^3 + 22x^2 - 12x + 8 = Ax(x - 2)(x + 2) + B(x - 2)(x + 2) + Cx^2(x + 2) + Dx^2(x - 2)$$

Hallamos A, B, C y D dando a x los valores 0, 2, –2 y 1:

$$\left.\begin{array}{lll} x = 0 & \Rightarrow \; 8 = -4B & \Rightarrow \; B = -2 \\ x = 2 & \Rightarrow \; 80 = 16C & \Rightarrow \; C = 5 \\ x = -2 & \Rightarrow \; 112 = -16D & \Rightarrow \; D = -7 \\ x = 1 & \Rightarrow \; 19 = -3A - 3B + 3C - D & \Rightarrow \; -3A = -9 \; \Rightarrow \; A = 3 \end{array}\right\}$$

Por tanto:

$$\int \frac{x^3 + 22x^2 - 12x + 8}{x^4 - 4x^2}\, dx = \int \left(\frac{3}{x} - \frac{2}{x^2} + \frac{5}{x - 2} - \frac{7}{x + 2} \right) dx =$$

$$= 3\, ln|x| + \frac{2}{x} + 5\, ln|x - 2| - 7\, ln|x + 2| + k$$

b) La fracción se puede simplificar:

$$\frac{x^3 - 4x^2 + 4x}{x^4 - 2x^3 - 4x^2 + 8x} = \frac{x(x - 2)^2}{x(x - 2)^2(x + 2)} = \frac{1}{x + 2}$$

$$\int \frac{x^3 - 4x^2 + 4x}{x^4 - 2x^3 - 4x^2 + 8x}\, dx = \int \frac{1}{x + 2}\, dx = ln|x + 2| + k$$

Ejercicio 12.-

Calcula las siguientes integrales inmediatas:

a) $\displaystyle\int (4x^2 - 5x + 7)\, dx$ **b)** $\displaystyle\int \frac{dx}{\sqrt[5]{x}}$ **c)** $\displaystyle\int \frac{1}{2x + 7}\, dx$ **d)** $\displaystyle\int (x - sen\, x)\, dx$

a) $\displaystyle\int (4x^2 - 5x + 7)\, dx = \frac{4x^3}{3} - \frac{5x^2}{2} + 7x + k$

b) $\displaystyle\int \frac{dx}{\sqrt[5]{x}} = \int x^{-1/5}\, dx = \frac{x^{4/5}}{4/5} + k = \frac{5\sqrt[5]{x^4}}{4} + k$

c) $\int \frac{1}{2x+7}\,dx = \frac{1}{2}\ ln|2x+7| + k$

d) $\int (x - sen\ x)dx = \frac{x^2}{2} + cox\ x + k$

Ejercicio 13.-

Resuelve estas integrales:

a) $\int (x^2+4x)(x^2-1)\,dx$ **b)** $\int (x-1)^3\,dx$

c) $\int \sqrt{3x}\,dx$ **d)** $\int (sen\ x + e^x)\,dx$

a) $\int (x^2+4x)(x^2-1)dx = \int (x^4+4x^3-x^2-4x)\,dx = \frac{x^5}{5} + x^4 - \frac{x^3}{3} - 2x^2 + k$

b) $\int (x-1)^3\,dx = \frac{(x-1)^4}{4} + k$

c) $\int \sqrt{3x}\,dx = \int \sqrt{3}\,x^{1/2}\,dx = \sqrt{3}\,\frac{x^{3/2}}{3/2} + k = \frac{2\sqrt{3x^3}}{3} + k$

d) $\int (sen\ x + e^x)\,dx = -cos\ x + e^x + k$

Ejercicio 14.-

Calcula las integrales siguientes:

a) $\int \sqrt[3]{\frac{x}{2}}\,dx$ **b)** $\int sen(x-4)\,dx$ **c)** $\int \frac{7}{cos^2 x}\,dx$ **d)** $\int (e^x + 3e^{-x})\,dx$

a) $\int \sqrt[3]{\frac{x}{2}}\,dx = \frac{1}{\sqrt[3]{2}} \int x^{1/3}\,dx = \frac{1}{\sqrt[3]{2}}\,\frac{x^{4/3}}{4/3} + k = \frac{3}{4}\sqrt[3]{\frac{x^4}{2}} + k$

b) $\int sen\,(x-4)dx = -cos\,(x-4) + k$

c) $\int \frac{7}{cos^2 x}\,dx = 7\ tg\ x + k$

d) $\int (e^x + 3e^{-x})dx = e^x - 3e^{-x} + k$

Ejercicio 15.-

Halla estas integrales:

a) $\int \frac{2}{x}\,dx$ **b)** $\int \frac{dx}{x-1}$ **c)** $\int \frac{x+\sqrt{x}}{x^2}\,dx$ **d)** $\int \frac{3}{1+x^2}\,dx$

a) $\int \frac{2}{x}\,dx = 2\ ln|x| + k$

b) $\int \frac{dx}{x-1} = ln|x-1| + k$

c) $\int \frac{x+\sqrt{x}}{x^2}\,dx = \int \left(\frac{1}{x} + x^{-3/2}\right) dx = ln|x| - \frac{2}{\sqrt{x}} + k$

d) $\int \frac{3}{1+x^2}\,dx = 3\ arc\ tg\ x + k$

Ejercicio 16.-

Resuelve las siguientes integrales:

a) $\int \frac{dx}{x-4}$ **b)** $\int \frac{dx}{(x-4)^2}$ **c)** $\int (x-4)^2\,dx$ **d)** $\int \frac{dx}{(x-4)^3}$

a) $\int \frac{dx}{x-4} = ln|x-4| + k$

b) $\int \frac{dx}{(x-4)^2} = \frac{-1}{(x-4)} + k$

c) $\int (x-4)^2 dx = \frac{(x-4)^3}{3} + k$

d) $\int \frac{dx}{(x-4)^3} = \int (x-4)^{-3}\,dx = \frac{(x-4)^{-2}}{-2} + k = \frac{-1}{2(x-4)^2} + k$

Ejercicio 17.-

Halla las siguientes integrales del tipo exponencial:

a) $\int e^{x-4}\,dx$ **b)** $\int e^{-2x+9}\,dx$ **c)** $\int e^{5x}\,dx$ **d)** $\int (3^x - x^3)\,dx$

a) $\int e^{x-4}\,dx = e^{x-4} + k$

b) $\int e^{-2x+9}\,dx = \frac{-1}{2}\int -2e^{-2x+9}\,dx = \frac{-1}{2}e^{-2x+9} + k$

c) $\int e^{5x}\,dx = \frac{1}{5}\int 5e^{5x}\,dx = \frac{1}{5}e^{5x} + k$

d) $\int (3^x - x^3)\,dx = \frac{3^x}{ln\ 3} - \frac{x^4}{4} + k$

Ejercicio 18.-

Resuelve las siguientes integrales del tipo arco tangente:

a) $\int \frac{dx}{4+x^2}$ **b)** $\int \frac{4\,dx}{3+x^2}$ **c)** $\int \frac{5\,dx}{4x^2+1}$ **d)** $\int \frac{2\,dx}{1+9x^2}$

a) $\int \frac{dx}{4+x^2} = \int \frac{1/4}{1+(x/2)^2}\,dx = \frac{1}{2}\int \frac{1/2}{1+(x/2)^2}\,dx = \frac{1}{2}\ arc\ tg\left(\frac{x}{2}\right) + k$

b) $\int \frac{4\,dx}{3+x^2} = \int \frac{4/3}{1+(x/\sqrt{3})^2}\,dx = \frac{4\sqrt{3}}{3}\int \frac{1/\sqrt{3}}{1+(x/\sqrt{3})^2}\,dx = \frac{4\sqrt{3}}{3}\,arc\ tg\left(\frac{x}{\sqrt{3}}\right) + k$

c) $\int \frac{5\,dx}{4x^2+1} = \frac{5}{2}\int \frac{2\,dx}{(2x)^2+1} = \frac{5}{2}\,arc\ tg\,(2x) + k$

d) $\int \frac{2\,dx}{1+9x^2} = \frac{2}{3}\int \frac{3\,dx}{1+(3x)^2} = \frac{2}{3}\,arc\ tg\,(3x) + k$

Ejercicio 19.-

Expresa las siguientes integrales de la forma:

$$\frac{\textbf{dividendo}}{\textbf{divisor}} = \textbf{cociente} + \frac{\textbf{resto}}{\textbf{divisor}}$$

y resuélvelas:

a) $\int \frac{x^2-5x+4}{x+1}\,dx$ **b)** $\int \frac{x^2+2x+4}{x+1}\,dx$ **c)** $\int \frac{x^3-3x^2+x-1}{x-2}\,dx$

a) $\int \frac{x^2-5x+4}{x+1}\,dx = \int \left(x - 6 + \frac{10}{x+1}\right)dx = \frac{x^2}{2} - 6x + 10\,ln|x+1| + k$

b) $\int \frac{x^2+2x+4}{x+1}\,dx = \int \left(x + 1 + \frac{3}{x+1}\right)dx = \frac{x^2}{2} + x + 3\,ln|x+1| + k$

c) $\int \frac{x^3-3x^2+x-1}{x-2}\,dx = \int \left(x^2 - x - 1 - \frac{3}{x-2}\right)dx =$

$= \frac{x^3}{3} - \frac{x^2}{2} - x - 3\,ln|x-2| + k$

Ejercicio 20.-

Halla estas integrales sabiendo que son del tipo arco seno:

a) $\int \frac{dx}{\sqrt{1-4x^2}}$ **b)** $\int \frac{dx}{\sqrt{4-x^2}}$ **c)** $\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx$ **d)** $\int \frac{dx}{x\sqrt{1-(ln\ x)^2}}$

a) $\int \frac{dx}{\sqrt{1-4x^2}} = \frac{1}{2}\int \frac{2\,dx}{\sqrt{1-(2x)^2}} = \frac{1}{2}\,arc\ sen\,(2x) + k$

b) $\int \frac{dx}{\sqrt{4-x^2}} = \int \frac{1/2\,dx}{\sqrt{1-(x/2)^2}} = arc\ sen\left(\frac{x}{2}\right) + k$

c) $\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx = \int \frac{e^x}{\sqrt{1-(e^x)^2}}\,dx = arc\ sen\,(e^x) + k$

d) $\int \frac{dx}{x\sqrt{1-(ln\ x)^2}} = \int \frac{1/x\,dx}{\sqrt{1-(ln\ x)^2}} = arc\ sen\,(ln|x|) + k$

Ejercicio 21.-

Resuelve las integrales siguientes,

a) $\int \cos x\, sen^3 x\, dx$ **b)** $\int 2x\, e^{x^2}\, dx$ **c)** $\int \frac{x\, dx}{(x^2+3)^5}$ **d)** $\int \frac{1}{x} ln^3 x\, dx$

a) $\int \cos x\, sen^3 x\, dx = \frac{sen^4 x}{4} + k$

b) $\int 2x\, e^{x^2}\, dx = e^{x^2} + k$

c) $\int \frac{x\, dx}{(x^2+3)^5} = \frac{1}{2}\int 2x(x^2+3)^{-5}\, dx = \frac{1}{2}\,\frac{(x^2+3)^{-4}}{-4} + k = \frac{-1}{8(x^2+3)^4} + k$

d) $\int \frac{1}{x} ln^3 x\, dx = \frac{ln^4|x|}{4} + k$

Ejercicio 22.-

Resuelve las siguientes integrales:

a) $\int x^4\, e^{x^5}\, dx$ **b)** $\int x\, sen\, x^2\, dx$ **c)** $\int \frac{dx}{\sqrt{9-x^2}}$ **d)** $\int \frac{x\, dx}{\sqrt{x^2+5}}$

a) $\int x^4\, e^{x^5}\, dx = \frac{1}{5}\int 5x^4\, e^{x^5}\, dx = \frac{1}{5} e^{x^5} + k$

b) $\int x\, sen\, x^2\, dx = \frac{1}{2}\int 2x\, sen\, x^2\, dx = \frac{-1}{2}\cos x^2 + k$

c) $\int \frac{dx}{\sqrt{9-x^2}} = \int \frac{1/3\, dx}{\sqrt{1-(x/3)^2}} = arc\, sen\left(\frac{x}{3}\right) + k$

d) $\int \frac{x\, dx}{\sqrt{x^2+5}} = \sqrt{x^2+5} + k$

Ejercicio 23.-

Resuelve las siguientes integrales:

a) $\int \sqrt{x^2-2x}\,(x-1)\,dx$ **b)** $\int tg\, x\, sec^2 x\, dx$

c) $\int \frac{(1+ln\, x)^2}{x}\, dx$ **d)** $\int \sqrt{(1+\cos x)^3}\, sen\, x\, dx$

a) $\int \sqrt{x^2-2x}\,(x-1)\, dx = \frac{1}{2}\int \sqrt{x^2-2x}\,(2x-2)\, dx = \frac{1}{2}\int (x^2-2x)^{1/2}(2x-2)\, dx =$

$= \frac{1}{2}\,\frac{(x^2-2x)^{3/2}}{3/2} + k = \frac{\sqrt{(x^2-2x)^3}}{3} + k$

b) $\int tg\, x\, sec^2 x\, dx = \frac{tg^2 x}{2} + k$

c) $\int \frac{(1+ln\, x)^2}{x}\, dx = \int (1+ln\, x)^2 \cdot \frac{1}{x}\, dx = \frac{(1+ln|x|)^3}{3} + k$

d) $\int \sqrt{(1+\cos x)^3}\, sen\, x\, dx = -\int (1+\cos x)^{3/2}(-sen\, x)\, dx = -\frac{(1+\cos x)^{5/2}}{5/2} + k =$

$= \frac{-2\sqrt{(1+\cos x)^5}}{5} + k$

Ejercicio 24.-

Aplica la integración por partes para resolver las siguientes integrales:

a) $\int x \ln x\, dx$ **b)** $\int e^x \cos x\, dx$ **c)** $\int x^2 \operatorname{sen} x\, dx$ **d)** $\int x^2 e^{2x}\, dx$

e) $\int \cos(\ln x)\, dx$ **f)** $\int x^2 \ln x\, dx$ **g)** $\int \operatorname{arc\,tg} x\, dx$ **h)** $\int (x+1)^2 e^x\, dx$

a) $\int x \ln x\, dx$

$$\begin{cases} u = \ln x \quad \rightarrow \quad du = \frac{1}{x}\, dx \\ dv = x\, dx \quad \rightarrow \quad v = \frac{x^2}{2} \end{cases}$$

$$\int x \ln x\, dx = \frac{x^2}{2} \ln x - \int \frac{x}{2}\, dx = \frac{x^2}{2} \ln|x| - \frac{x^2}{4} + k$$

b) $\int e^x \cos x\, dx$

$$\begin{cases} u = e^x \quad \rightarrow \quad du = e^x\, dx \\ dv = \cos x\, dx \quad \rightarrow \quad v = \operatorname{sen} x \end{cases}$$

$$\int e^x \cos x\, dx = e^x \operatorname{sen} x - \underbrace{\int e^x \operatorname{sen} x\, dx}_{I_1}$$

$$\begin{cases} u_1 = e^x \quad \rightarrow \quad du_1 = e^x\, dx \\ dv_1 = \operatorname{sen} x\, dx \quad \rightarrow \quad v_1 = -\cos x \end{cases}$$

$$I_1 = -e^x \cos x + \int e^x \cos x\, dx$$

Por tanto:

$$\int e^x \cos x\, dx = e^x \operatorname{sen} x + e^x \cos x - \int e^x \cos x\, dx$$

$$2\int e^x \cos x\, dx = e^x \operatorname{sen} x + e^x \cos x$$

$$\int e^x \cos x\, dx = \frac{e^x \operatorname{sen} x + e^x \cos x}{2} + k$$

c) $\int x^2 \, sen\, x \, dx$

$$\begin{cases} u = x^2 \quad \rightarrow \quad du = 2x\, dx \\ dv = sen\, x\, dx \quad \rightarrow \quad v = -cos\, x \end{cases}$$

$$\int x^2\, sen\, x\, dx = -x^2\, cos\, x + \int 2x\, cos\, x\, dx = -x^2\, cos\, x + 2\underbrace{\int x\, cos\, x\, dx}_{I_1}$$

$$\begin{cases} u_1 = x \quad \rightarrow \quad du_1 = dx \\ dv_1 = cos\, x\, dx \quad \rightarrow \quad v_1 = sen\, x \end{cases}$$

$$I_1 = x\, sen\, x - \int sen\, x\, dx = x\, sen\, x + cos\, x$$

Por tanto:

$$\int x^2\, sen\, x\, dx = -x^2\, cos\, x + 2x\, sen\, x + 2\, cos\, x + k$$

d) $\int x^2\, e^{2x}\, dx$

$$\begin{cases} u = x^2 \quad \rightarrow \quad du = 2x\, dx \\ dv = e^{2x}\, dx \quad \rightarrow \quad v = \frac{1}{2}\, e^{2x} \end{cases}$$

$$\int x^2\, e^{2x}\, dx = \frac{x^2}{2}\, e^{2x} - \underbrace{\int x\, e^{2x}\, dx}_{I_1}$$

$$\begin{cases} u_1 = x \quad \rightarrow \quad du_1 = dx \\ dv_1 = e^{2x}\, dx \quad \rightarrow \quad v_1 = \frac{1}{2}\, e^{2x} \end{cases}$$

$$I_1 = \frac{x}{2}\, e^{2x} - \int \frac{1}{2}\, e^{2x}\, dx = \frac{x}{2}\, e^{2x} - \frac{1}{4}\, e^{2x}$$

Por tanto: $\int x^2\, e^{2x}\, dx = \frac{x^2}{2}\, e^{2x} - \frac{x}{2}\, e^{2x} + \frac{1}{4}\, e^{2x} + k = \left(\frac{x^2}{2} - \frac{x}{2} + \frac{1}{4}\right) e^{2x} + k$

e) $\int cos(ln\,x)\,dx$

$$\begin{cases} u = cos\,(ln\,x) \quad \rightarrow \quad du = -sen\,(ln\,x) \cdot \dfrac{1}{x}\,dx \\ dv = dx \quad \rightarrow \quad v = x \end{cases}$$

$$\int cos\,(ln\,x)\,dx = x\,cos\,(ln\,x) + \underbrace{\int sen\,(ln\,x)\,dx}_{I_1}$$

$$\begin{cases} u_1 = sen\,(ln\,x) \quad \rightarrow \quad du_1 = cos\,(ln\,x) \cdot \dfrac{1}{x}\,dx \\ dv_1 = dx \quad \rightarrow \quad v_1 = x \end{cases}$$

$$I_1 = x\,sen\,(ln\,x) - \int cos\,(ln\,x)\,dx$$

Por tanto:

$$\int cos\,(ln\,x)\,dx = x\,cos\,(ln\,x) + x\,sen\,(ln\,x) - \int cos\,(ln\,x)\,dx$$

$$2\int cos\,(ln\,x)\,dx = x\,cos\,(ln\,x) + x\,sen\,(ln\,x)$$

$$\int cos\,(ln\,x)\,dx = \frac{x\,cos\,(ln\,x) + x\,sen\,(ln\,x)}{2} + k$$

f) $\int x^2\,ln\,x\,dx$

$$\begin{cases} u = ln\,x \quad \rightarrow \quad du = \dfrac{1}{x}\,dx \\ dv = x^2\,dx \quad \rightarrow \quad v = \dfrac{x^3}{2} \end{cases}$$

$$\int x^2\,ln\,x\,dx = \frac{x^3\,ln\,x}{3} - \int \frac{x^2}{3}\,dx = \frac{x^3\,ln\,x}{3} - \frac{x^3}{9} + k$$

g) $\int arc\ tg\ x\ dx$

$$\begin{cases} u = arc\ tg\ x \rightarrow du = \dfrac{1}{1+x^2}\,dx \\ dv = dx \rightarrow v = x \end{cases}$$

$$\int arc\ tg\ x = x\ arc\ tg\ x - \int \frac{1}{1+x^2}\,dx = x\ arc\ tg\ x - \frac{1}{2}\int \frac{2x}{1+x^2}\,dx =$$

$$= x\ arc\ tg\ x - \frac{1}{2}\ln(1+x^2) + k$$

h) $\int (x+1)^2 e^x\,dx$

$$\begin{cases} u = (x+1)^2 \rightarrow du = 2(x+1)\,dx \\ dv = e^x\,dx \rightarrow v = e^x \end{cases}$$

$$\int (x+1)^2 e^x\,dx = (x+1)^2 e^x - 2\underbrace{\int (x+1)\,e^x\,dx}_{I_1}$$

$$\begin{cases} u_1 = (x+1) \rightarrow du_1 = dx \\ dv_1 = e^x\,dx \rightarrow v_1 = e^x \end{cases}$$

$$I_1 = (x+1)\,e^x - \int e^x\,dx = (x+1)\,e^x - e^x = (x+1-1)\,e^x = x\,e^x$$

Por tanto:

$$\int (x+1)^2 e^x\,dx = (x+1)^2 e^x - 2x\,e^x + k =$$

$$= (x^2 + 2x + 1 - 2x)\,e^x + k = (x^2+1)\,e^x + k$$

Ejercicio 25.-

Determina el valor de las integrales que se proponen a continuación:

a) $\int x \cdot 2^{-x}\, dx$ **b)** $\int arc\cos x\, dx$ **c)** $\int x \cos 3x\, dx$ **d)** $\int x^5 e^{-x^3}\, dx$

a) $\int x \cdot 2^{-x}\, dx$

$$\begin{cases} u = x \rightarrow du = dx \\ dv = 2^{-x}\, dx \rightarrow v = \dfrac{-2^{-x}}{ln\, 2} \end{cases}$$

$$\int x\, 2^{-x}\, dx = \frac{-x \cdot 2^{-x}}{ln\, 2} + \int \frac{2^{-x}}{ln\, 2}\, dx = \frac{-x \cdot 2^{-x}}{ln\, 2} + \frac{1}{ln\, 2}\int 2^{-x}\, dx =$$

$$= \frac{-x \cdot 2^{-x}}{ln\, 2} - \frac{2^{-x}}{(ln\, 2)^2} + k$$

b) $\int arc\cos x\, dx$

$$\begin{cases} u = arc\cos x \rightarrow du = \dfrac{-1}{\sqrt{1 - x^2}}\, dx \\ dv = dx \rightarrow v = x \end{cases}$$

$$\int arc\cos x\, dx = x\, arc\cos x - \int \frac{-x}{\sqrt{1 - x^2}}\, dx = x\, arc\cos x - \sqrt{1 - x^2} + k$$

c) $\int x \cos 3x\, dx$

$$\begin{cases} u = x \rightarrow du = dx \\ dv = \cos 3x\, dx \rightarrow v = \dfrac{1}{3}\, sen\, 3x \end{cases}$$

$$\int x \cos 3x\, dx = \frac{x}{3}\, sen\, 3x - \frac{1}{3}\int sen\, 3x\, dx = \frac{x}{3}\, sen\, 3x + \frac{1}{9}\cos 3x + k$$

d) $\int x^5 e^{-x^3}\, dx = \int \underbrace{x^3}_{u} \cdot \underbrace{x^2 e^{-x^3}\, dx}_{dv}$

$$\begin{cases} u = x^3 \rightarrow du = 3x^2\, dx \\ dv = x^2 e^{-x^3}\, dx \rightarrow v = \dfrac{-1}{3}\, e^{-x^3} \end{cases}$$

$$\int x^5 e^{-x^3}\, dx = \frac{-x^3}{3}\, e^{-x^3} + \int x^2 e^{-x^3}\, dx = \frac{-x^3}{3}\, e^{-x^3} - \frac{1}{3}\, e^{-x^3} + k =$$

$$= \frac{(-x^3 - 1)}{3}\, e^{-x^3} + k$$

Ejercicio 26.-

Resuelve las siguientes integrales:

a) $\displaystyle\int \frac{2x-4}{(x-1)^2\,(x+3)}\,dx$ **b)** $\displaystyle\int \frac{2x+3}{(x-2)\,(x+5)}\,dx$

c) $\displaystyle\int \frac{1}{(x-1)\,(x+3)^2}\,dx$ **d)** $\displaystyle\int \frac{3x-2}{x^2-4}\,dx$

a) $\displaystyle\int \frac{2x-4}{(x-1)^2\,(x+3)}\,dx$

Descomponemos en fracciones simples:

$$\frac{2x-4}{(x-1)^2\,(x+3)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+3}$$

$$\frac{2x-4}{(x-1)^2\,(x+3)} = \frac{A(x-1)(x+3) + B(x+3) + C(x-1)^2}{(x-1)^2\,(x+3)}$$

$$2x-4 = A(x-1)(x+3) + B(x+3) + C(x-1)^2$$

Hallamos A, B y C:

$$\left.\begin{array}{llll} x = 1 & \rightarrow\ -2 = 4B & \rightarrow & B = -1/2 \\ x = -3 & \rightarrow\ -10 = 16C & \rightarrow & C = -5/8 \\ x = 0 & \rightarrow\ -4 = -3A + 3B + C & \rightarrow & A = 5/8 \end{array}\right\}$$

Por tanto:

$$\int \frac{2x-4}{(x-1)^2\,(x+3)}\,dx = \int \frac{5/8}{x-1}\,dx + \int \frac{-1/2}{(x-1)^2}\,dx + \int \frac{-5/8}{x+3}\,dx =$$

$$= \frac{5}{8}\,ln|x-1| + \frac{1}{2}\cdot\frac{1}{(x-1)} - \frac{5}{8}\,ln|x+3| + k = \frac{5}{8}\,ln\left|\frac{x-1}{x+3}\right| + \frac{1}{2x-2} + k$$

b) $\displaystyle\int \frac{2x+3}{(x-2)\,(x+5)}\,dx$

Descomponemos en fracciones simples:

$$\frac{2x+3}{(x-2)\,(x+5)} = \frac{A}{x-2} + \frac{B}{x+5} = \frac{A(x+5) + B(x-2)}{(x-2)\,(x+5)}$$

$$2x+3 = A(x+5) + B(x-2)$$

Hallamos A y B:

$$\left.\begin{array}{llll} x = 2 & \rightarrow\ 7 = 7A & \rightarrow & A = 1 \\ x = -5 & \rightarrow\ -7 = -7B & \rightarrow & B = 1 \end{array}\right\}$$

Por tanto:

$$\int \frac{2x+3}{(x-2)\,(x+5)}\,dx = \int \frac{1}{x-2}\,dx + \int \frac{1}{x+5}\,dx =$$

$$= ln|x-2| + ln|x+5| + k = ln|(x-2)(x+5)| + k$$

c) $\int \frac{1}{(x-1)(x+3)^2}\,dx$

Descomponemos en fracciones simples:

$$\frac{1}{(x-1)(x+3)^2} = \frac{A}{x-1} + \frac{B}{x+3} + \frac{C}{(x+3)^2}$$

$$\frac{1}{(x-1)(x+3)^2} = \frac{A(x+3)^2 + B(x-1)(x+3) + C(x-1)}{(x-1)(x+3)^2}$$

$$1 = A(x+3)^2 + B(x-1)(x+3) + C(x-1)$$

Hallamos A, B y C:

$$\left.\begin{array}{llll} x = 1 & \rightarrow\ 1 = 16A & \rightarrow & A = 1/16 \\ x = -3 & \rightarrow\ 1 = -4C & \rightarrow & C = -1/4 \\ x = 0 & \rightarrow\ 1 = 9A - 3B - C & \rightarrow & B = -1/16 \end{array}\right\}$$

Por tanto:

$$\int \frac{1}{(x-1)(x+3)^2}\,dx = \int \frac{1/16}{x-1}\,dx + \int \frac{-1/16}{x+3}\,dx + \int \frac{-1/4}{(x+3)^2}\,dx =$$

$$= \frac{1}{16}\,ln|x-1| - \frac{1}{16}\,ln|x+3| + \frac{1}{4}\cdot\frac{1}{(x+3)} + k =$$

$$= \frac{1}{16}\,ln\left|\frac{x-1}{x+3}\right| + \frac{1}{4(x+3)} + k$$

d) $\int \frac{3x-2}{x^2-4}\,dx = \int \frac{3x-2}{(x-2)(x+2)}\,dx$

Descomponemos en fracciones simples:

$$\frac{3x-2}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2} = \frac{A(x+2) + B(x-2)}{(x-2)(x+2)}$$

$$3x - 2 = A(x+2) + B(x-2)$$

Hallamos A y B:

$$\left.\begin{array}{llll} x = 2 & \rightarrow\ 4 = 4A & \rightarrow & A = 1 \\ x = -2 & \rightarrow\ -8 = -4B & \rightarrow & B = 2 \end{array}\right\}$$

Por tanto:

$$\int \frac{3x-2}{x^2-4}\,dx = \int \frac{1}{x-2}\,dx + \int \frac{2}{x+2}\,dx =$$

$$= ln|x-2| + 2\,ln|x+2| + k = ln\left[|x-2|(x+2)^2\right] + k$$

Ejercicio 27.-

Resuelve las integrales:

a) $\int \frac{\ln x}{x}\, dx$ **b)** $\int \frac{1 - \operatorname{sen} x}{x + \cos x}\, dx$ **c)** $\int \frac{1}{x \ln x}\, dx$

d) $\int \frac{1 + e^x}{e^x + x}\, dx$ **e)** $\int \frac{\operatorname{sen}(1/x)}{x^2}\, dx$ **f)** $\int \frac{2x - 3}{x + 2}\, dx$

g) $\int \frac{\operatorname{arc\,tg} x}{1 + x^2}\, dx$ **h)** $\int \frac{\operatorname{sen} x}{\cos^4 x}\, dx$

a) $\int \frac{\ln x}{x}\, dx = \int \frac{1}{x} \ln x\, dx = \frac{\ln^2|x|}{2} + k$

b) $\int \frac{1 - \operatorname{sen} x}{x + \cos x}\, dx = \ln|x + \cos x| + k$

c) $\int \frac{1}{x \ln x}\, dx = \int \frac{1/x}{\ln x}\, dx = \ln|\ln|x|| + k$

d) $\int \frac{1 + e^x}{e^x + x}\, dx = \ln|e^x + x| + k$

e) $\int \frac{\operatorname{sen}(1/x)}{x^2}\, dx = -\int \frac{-1}{x^2} \operatorname{sen}\left(\frac{1}{x}\right) dx = \cos\left(\frac{1}{x}\right) + k$

f) $\int \frac{2x - 3}{x + 2}\, dx = \int \left(2 - \frac{7}{x + 2}\right) dx = 2x - 7 \ln|x + 2| + k$

g) $\int \frac{\operatorname{arc\,tg} x}{1 + x^2}\, dx = \int \frac{1}{1 + x^2} \operatorname{arc\,tg} x\, dx = \frac{\operatorname{arc\,tg}^2 x}{2} + k$

h) $\int \frac{\operatorname{sen} x}{\cos^4 x}\, dx = -\int (-\operatorname{sen} x)(\cos x)^{-4}\, dx = \frac{-(\cos x)^{-3}}{-3} + k = \frac{1}{3 \cos^3 x} + k$

Ejercicio 28.-

Resuelve por sustitución:

a) $\int x\sqrt{x+1}\,dx$ b) $\int \frac{dx}{x-\sqrt[4]{x}}$ c) $\int \frac{x}{\sqrt{x+1}}\,dx$

d) $\int \frac{1}{x\sqrt{x+1}}\,dx$ e) $\int \frac{1}{x+\sqrt{x}}\,dx$ f) $\int \frac{\sqrt{x}}{1+x}\,dx$

☛ ***a) Haz*** $x+1=t^2$. ***b) Haz*** $x=t^4$.

a) $\int x\sqrt{x+1}\,dx$

Cambio: $x+2=t^2 \rightarrow dx=2t\,dt$

$$\int x\sqrt{x+1}\,dx=\int (t^2-1)t\cdot 2t\,dt=\int (2t^4-2t^2)\,dt=\frac{2t^5}{5}-\frac{2t^3}{3}+k=$$

$$=\frac{2\sqrt{(x+1)^5}}{5}-\frac{2\sqrt{(x+1)^3}}{3}+k$$

b) $\int \frac{dx}{x-\sqrt[4]{x}}$

Cambio: $x=t^4 \rightarrow dx=4t^3\,dt$

$$\int \frac{dx}{x-\sqrt[4]{x}}=\int \frac{4t^3\,dt}{t^4-t}=\int \frac{4t^2\,dt}{t^3-1}=\frac{4}{3}\int \frac{3t^2\,dt}{t^3-1}=\frac{4}{3}\,ln|t^3-1|+k=$$

$$=\frac{4}{3}\,ln|\sqrt[4]{x^3}-1|+k$$

c) $\int \frac{x}{\sqrt{x+1}}\,dx$

Cambio: $x+1=t^2 \rightarrow dx=2t\,dt$

$$\int \frac{x}{\sqrt{x+1}}\,dx=\int \frac{(t^2-1)}{t}\cdot 2t\,dt=\int (2t^2-2)\,dt=\frac{2t^3}{3}-2t+k=$$

$$=\frac{2\sqrt{(x+1)^3}}{3}-2\sqrt{x+1}+k$$

d) $\int \frac{1}{x\sqrt{x+1}}\,dx$

Cambio: $x+1=t^2 \rightarrow dx=2t\,dt$

$$\int \frac{1}{x\sqrt{x+1}}\,dx=\int \frac{2t\,dt}{(t^2-1)t}=\int \frac{2\,dt}{(t+1)(t-1)}$$

Descomponemos en fracciones simples:

$$\frac{2}{(t+1)(t-1)}=\frac{A}{t+1}+\frac{B}{t-1}=\frac{A(t-1)+B(t+1)}{(t+1)(t-1)}$$

$2 = A(t-1) + B(t+1)$

Hallamos A y B:

$$\left.\begin{array}{lllll} t=-1 & \rightarrow & 2=-2A & \rightarrow & A=-1 \\ t=1 & \rightarrow & 2=2B & \rightarrow & B=1 \end{array}\right\}$$

Por tanto:

$$\int \frac{2\,dt}{(t+1)(t-1)} = \int\left(\frac{-1}{t+1} + \frac{1}{t-1}\right) dt = -ln|t+1| + ln|t-1| + k =$$

$$= ln\left|\frac{t-1}{t+1}\right| + k$$

Así:

$$\int \frac{1}{x\ \sqrt{x+1}}\, dx = ln\left|\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}\right| + k$$

e) $\int \frac{1}{x+\sqrt{x}}\, dx$

Cambio: $x = t^2 \rightarrow dx = 2t\, dt$

$$\int \frac{1}{x+\sqrt{x}}\, dx = \int \frac{2t\, dt}{t^2+t} = \int \frac{2\, dt}{t+1} = 2\ ln|t+1| + k =$$

$$= 2\ ln\,(\sqrt{x}+1) + k$$

f) $\int \frac{\sqrt{x}}{1+x}\, dx$

Cambio: $x = t^2 \rightarrow dx = 2t\, dt$

$$\int \frac{\sqrt{x}}{1+x}\, dx = \int \frac{t \cdot 2t\, dt}{1+t^2} = \int \frac{2t^2\, dt}{1+t^2} = \int\left(2 - \frac{2}{1+t^2}\right) dt =$$

$$= 2t - 2\ arc\ tg\ t + k = 2\sqrt{x} - 2\ arc\ tg\ \sqrt{x} + k$$

Ejercicio 29.-

Resuelve, utilizando un cambio de variable, estas integrales:

a) $\int \sqrt{9-4x^2}\, dx$ **b)** $\int \frac{dx}{e^{2x}-3e^x}$ **c)** $\int \frac{e^{3x}-e^x}{e^{2x}+1}\, dx$ **d)** $\int \frac{1}{1+\sqrt{x}}\, dx$

☛ ***a) Haz*** $sen\ t = 2x/3$.

a) $\int \sqrt{9-4x^2}\, dx$

Cambio: $sen\ t = \frac{2x}{3} \rightarrow x = \frac{3}{2}\, sen\ t \rightarrow dx = \frac{3}{2}\, cos\ t\, dt$

$$\int \sqrt{9-4x^2}\, dx = \int \sqrt{9-4\cdot \frac{9}{4}\, sen^2\, t}\ \cdot \frac{3}{2}\, cos\, t\, dt = \int 3\, cos\, t \cdot \frac{3}{2}\, cos\, t\, dt =$$

$$= \frac{9}{2} \int cos^2\, t\, dt = \frac{9}{2} \int \left(\frac{1}{2} - \frac{cos\, 2t}{2} \right) dt = \frac{9}{2} \left(\frac{1}{2} t + \frac{1}{4}\, sen\, 2t \right) + k =$$

$$= \frac{9}{4} t + \frac{9}{8}\, sen\, 2t + k = \frac{9}{4}\, arc\, sen \left(\frac{2x}{3} \right) + \frac{9}{8} \cdot 2\, sen\, t\, cos\, t + k =$$

$$= \frac{9}{4}\, arc\, sen \left(\frac{2x}{3} \right) + \frac{9}{4} \cdot \frac{2x}{3} \sqrt{1 - \frac{4x^2}{9}} + k =$$

$$= \frac{9}{4}\, arc\, sen \left(\frac{2x}{3} \right) + \frac{x}{2} \cdot \sqrt{9-4x^2} + k$$

b) $\int \frac{dx}{e^{2x} - 3e^x}$

Cambio: $e^x = t \;\rightarrow\; x = ln\, t \;\rightarrow\; dx = \frac{1}{t}\, dt$

$$\int \frac{dx}{e^{2x} - 3e^x} = \int \frac{1/t}{t^2 - 3t}\, dt = \int \frac{1}{t^3 - 3t^2}\, dt = \int \frac{1}{t^2(t-3)}\, dt$$

Descomponemos en fracciones simples:

$$\frac{1}{t^2(t-3)} = \frac{A}{t} + \frac{B}{t^2} + \frac{C}{t-3} = \frac{At(t-3) + B(t-3) + Ct^2}{t^2(t-3)}$$

$$1 = At(t-3) + B(t-3) + Ct^2$$

Hallamos A, B y C:

$$\left. \begin{array}{llll} t = 0 & \rightarrow\; 1 = -3B & \rightarrow & B = -1/3 \\ t = 3 & \rightarrow\; 1 = 9C & \rightarrow & C = 1/9 \\ t = 1 & \rightarrow\; 1 = -2A - 2B + C & \rightarrow & A = -1/9 \end{array} \right\}$$

Así, tenemos que:

$$\int \frac{1}{t^2(t-3)}\, dt = \int \left(\frac{-1/9}{t} + \frac{-1/3}{t^2} + \frac{1/9}{t-3} \right) dt =$$

$$= \frac{-1}{9}\, ln|t| + \frac{1}{3t} + \frac{1}{9}\, ln|t-3| + k$$

Por tanto:

$$\int \frac{dx}{e^{2x} - 3e^x} = \frac{-1}{9}\, ln\, e^x + \frac{1}{3e^x} + \frac{1}{9}\, ln|e^x - 3| + k =$$

$$= -\frac{1}{9} x + \frac{1}{3e^x} + \frac{1}{9}\, ln|e^x - 3| + k$$

c) $\int \frac{e^{3x} - e^{x}}{e^{2x} + 1} \, dx$

Cambio: $e^x = t \;\rightarrow\; x = \ln t \;\rightarrow\; dx = \frac{1}{t}\, dt$

$$\int \frac{e^{3x} - e^{x}}{e^{2x} + 1} \, dx = \int \frac{t^3 - t}{t^2 + 1} \cdot \frac{1}{t} \, dt = \int \frac{t^2 - 1}{t^2 + 1} \, dt = \int \left(1 - \frac{2}{t^2 + 1}\right) dt =$$

$$= t - 2 \, arc\, tg\, t + k = e^x - 2 \, arc\, tg\, (e^x) + k$$

d) $\int \frac{1}{1 + \sqrt{x}} \, dx$

Cambio: $x = t^2 \;\rightarrow\; dx = 2t\, dt$

$$\int \frac{1}{1 + \sqrt{x}} \, dx = \int \frac{2t\, dt}{1 + t} = \int \left(2 - \frac{2}{1 + t}\right) dt = 2t - 2 \ln|1 + t| + k =$$

$$= 2\sqrt{x} - 2 \ln(1 + \sqrt{x}) + k$$

Ejercicio 30.-

Encuentra la primitiva de $f(x) = \frac{1}{1 + 3x}$ que se anula para $x = 0$.

$$F(x) = \int \frac{1}{1 + 3x} \, dx = \frac{1}{3} \int \frac{3}{1 + 3x} \, dx = \frac{1}{3} \ln|1 + 3x| + k$$

$$F(0) = k = 0$$

Por tanto: $F(x) = \frac{-1}{3} \ln|1 + 3x|$

Ejercicio 31.-

Halla la función F para la que $F'(x) = \frac{1}{x^2}$ y $F(1) = 2$.

$$F(x) = \int \frac{1}{x^2} \, dx = \frac{-1}{x} + k$$

$$F(1) = -1 + k = 2 \;\Rightarrow\; k = 3$$

Por tanto: $F(x) = \frac{-1}{x} + 3$

Ejercicio 32.-

De todas las primitivas de la función $y = 4x - 6$, ¿cuál de ellas toma el valor 4 para $x = 1$?

$$F(x) = \int (4x - 6) \, dx = 2x^2 - 6x + k$$

$$F(1) = 2 - 6 + k = 4 \;\Rightarrow\; k = 8$$

Por tanto: $F(x) = 2x^2 - 6x + 8$

Ejercicio 33.-

Halla $f(x)$ sabiendo que $f''(x) = 6x$, $f'(0) = 1$ y $f(2) = 5$.

$$\left.\begin{array}{l} f'(x) = \int 6x\,dx = 3x^2 + c \\ f'(0) = c = 1 \end{array}\right\} \quad f'(x) = 3x^2 + 1$$

$$\left.\begin{array}{l} f(x) = \int (3x^2 + 1)\,dx = x^3 + x + k \\ f(2) = 10 + k = 5 \quad \Rightarrow \quad k = -5 \end{array}\right\}$$

Por tanto: $f(x) = x^3 + x - 5$

Ejercicio 34.-

Resuelve las siguientes integrales por sustitución:

a) $\displaystyle\int \frac{e^x}{1-\sqrt{e^x}}\,dx$ **b)** $\displaystyle\int \sqrt{e^x - 1}\,dx$

☛ ***a) Haz*** $\sqrt{e^x} = t$. ***b) Haz*** $\sqrt{e^x - 1} = t$.

a) $\displaystyle\int \frac{e^x}{1-\sqrt{e^x}}\,dx$

Cambio: $\sqrt{e^x} = t \;\rightarrow\; e^{x/2} = t \;\rightarrow\; \dfrac{x}{2} = \ln t \;\rightarrow\; dx = \dfrac{2}{t}\,dt$

$$\int \frac{e^x}{1-\sqrt{e^x}} = \int \frac{t^2 \cdot (2/t)\,dt}{1-t} = \int \frac{2t\,dt}{1-t} = \int \left(-2 + \frac{2}{1-t}\right) dt =$$

$$= -2t - 2\ln|1-t| + k = -2\sqrt{e^x} - 2\ln|1-\sqrt{e^x}| + k$$

b) $\displaystyle\int \sqrt{e^x - 1}\,dx$

Cambio: $\sqrt{e^x - 1} = t \;\rightarrow\; e^x = t^2 + 1 \;\rightarrow\; x = \ln(t^2+1) \;\rightarrow\; dx = \dfrac{2t}{t^2+1}\,dt$

$$\int \sqrt{e^x - 1}\,dx = \int t \cdot \frac{2t}{t^2+1}\,dt = \int \frac{2t^2}{t^2+1}\,dt = \int \left(2 - \frac{2}{t^2+1}\right) dt =$$

$$= 2t - 2\,arc\,tg\,t + k = 2\sqrt{e^x - 1} - 2\,arc\,tg\,\sqrt{e^x - 1} + k$$

Ejercicio 35.-

Determina la función $f(x)$ sabiendo que:

$$f''(x) = x \ln x, \; f'(1) = 0 \;\; \text{y} \;\; f(e) = \frac{e}{4}$$

$f'(x) = \displaystyle\int x \ln x\,dx$

Integramos por partes:

$$\begin{cases} u = \ln x \quad \rightarrow \quad du = \dfrac{1}{x}\,dx \\ dv = x\,dx \quad \rightarrow \quad v = \dfrac{x^2}{2} \end{cases}$$

$$\left. \begin{array}{l} f'(x) = \dfrac{x^2}{2}\ln x - \displaystyle\int \dfrac{x}{2}\,dx = \dfrac{x^2}{2}\ln x - \dfrac{x^2}{4} + k = \dfrac{x^2}{2}\left(\ln x - \dfrac{1}{2}\right) + k \\ f'(1) = \dfrac{1}{2}\left(-\dfrac{1}{2}\right) + k = -\dfrac{1}{4} + k = 0 \quad \Rightarrow \quad k = \dfrac{1}{4} \end{array} \right\}$$

$$f'(x) = \frac{x^2}{2}\left(\ln x - \frac{1}{2}\right) + \frac{1}{4}$$

$$f(x) = \int\left[\frac{x^2}{2}\left(\ln x - \frac{1}{2}\right) + \frac{1}{4}\right]dx = \underbrace{\int \frac{x^2}{2}\left(\ln x - \frac{1}{2}\right)dx}_{I} + \frac{1}{4}x$$

$$\begin{cases} u = \left(\ln x - \dfrac{1}{2}\right) \quad \rightarrow \quad du = \dfrac{1}{x}\,dx \\ dv = \dfrac{x^2}{2}\,dx \quad \rightarrow \quad v = \dfrac{x^3}{6} \end{cases}$$

$$I = \frac{x^3}{6}\left(\ln x - \frac{1}{2}\right) - \int \frac{x^2}{6}\,dx = \frac{x^3}{6}\left(\ln x - \frac{1}{2}\right) - \frac{x^3}{18} + k$$

Por tanto:

$$\left. \begin{array}{l} f(x) = \dfrac{x^3}{6}\left(\ln x - \dfrac{1}{2}\right) - \dfrac{x^3}{18} + \dfrac{1}{4}x + k \\ f(e) = \dfrac{e^3}{12} - \dfrac{e^3}{18} + \dfrac{e}{4} + k = \dfrac{e^3}{36} + \dfrac{e}{4} + k = \dfrac{e}{4} \quad \Rightarrow \quad k = -\dfrac{e^3}{36} \end{array} \right\}$$

$$f(x) = \frac{x^3}{6}\left(\ln x - \frac{1}{2}\right) - \frac{x^3}{18} + \frac{1}{4}x - \frac{e^3}{36}$$

Ejercicio 36.-

Calcula la expresión de una función $f(x)$ tal que $f'(x) = x\,e^{-x^2}$ y que $f(0) = \dfrac{1}{2}$.

$$f(x) = \int x\,e^{-x^2}\,dx = -\frac{1}{2}\int -2x\,e^{-x^2}\,dx = -\frac{1}{2}\,e^{-x^2} + k$$

$$f(0) = -\frac{1}{2} + k = \frac{1}{2} \quad \Rightarrow \quad k = 1$$

Por tanto: $f(x) = -\dfrac{1}{2}\,e^{-x^2} + 1$

Ejercicio 37.-

Encuentra la función derivable $f: [-1, 1] \to \mathbb{R}$ que cumple $f(1) = -1$ y

$$f'(x) = \begin{cases} x^2 - 2x & \text{si } -1 \le x < 0 \\ e^x - 1 & \text{si } 0 \le x \le 1 \end{cases}$$

- Si $x \neq 0$:

$$f(x) = \begin{cases} \dfrac{x^3}{3} - x^2 + k & \text{si } -1 \le x < 0 \\ e^x - x + c & \text{si } 0 < x \le 1 \end{cases}$$

- Hallamos k y c teniendo en cuenta que $f(1) = -1$ y que $f(x)$ ha de ser continua en $x = 0$.

$$f(1) = -1 \Rightarrow e - 1 + c = -1 \Rightarrow c = -e$$

$$\left.\begin{array}{l} \lim\limits_{x \to 0^-} f(x) = k \\ \lim\limits_{x \to 0^+} f(x) = 1 - e \end{array}\right\} k = 1 - e$$

Por tanto: $f(x) = \begin{cases} \dfrac{x^3}{3} - x^2 + 1 - e & \text{si } -1 \le x < 0 \\ e^x - x - e & \text{si } 0 \le x \le 1 \end{cases}$

Ejercicio 38.-

De una función derivable se sabe que pasa por el punto $A(-1, -4)$ y que su derivada es:

$$f'(x) = \begin{cases} 2 - x & \text{si } x \le 1 \\ 1/x & \text{si } x > 1 \end{cases}$$

a) Halla la expresión de $f(x)$.

b) Obtén la ecuación de la recta tangente a $f(x)$ en $x = 2$.

a) Si $x \neq 1$:

$$f(x) = \begin{cases} 2x - \dfrac{x^2}{2} + k & \text{si } x < 1 \\ ln\, x + c & \text{si } x > 1 \end{cases}$$

Hallamos k y c teniendo en cuenta que $f(-1) = -4$ y que $f(x)$ ha de ser con tinua en $x = 1$.

$$f(-1) = -\frac{5}{2} + k = -4 \Rightarrow k = -\frac{3}{2}$$

$$\left.\begin{array}{l} \lim\limits_{x \to 1^-} f(x) = \dfrac{3}{2} - \dfrac{3}{2} = 0 \\ \lim\limits_{x \to 1^+} f(x) = c \end{array}\right\} c = 0$$

Por tanto: $f(x) = \begin{cases} 2x - \dfrac{x^2}{2} - \dfrac{3}{2} & \text{si } x < 1 \\ ln\, x & \text{si } x \ge 1 \end{cases}$

b) $f(2) = ln\, 2$; $f'(2) = \dfrac{1}{2}$

La ecuación de la recta tangente será: $y = ln\, 2 + \dfrac{1}{2}(x - 2)$