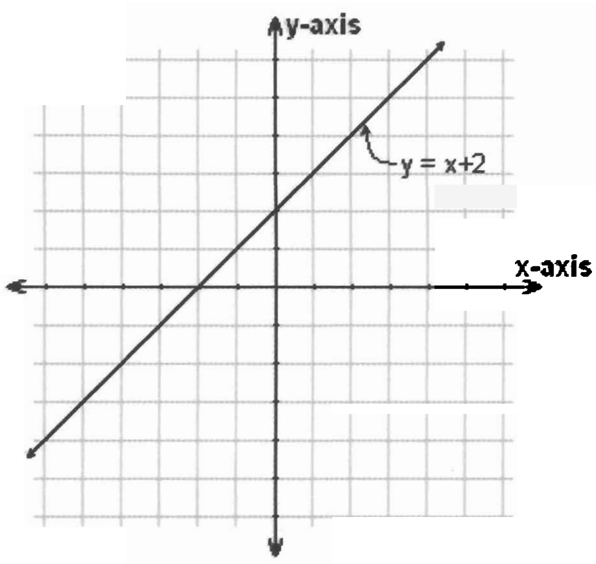
Linear Functions

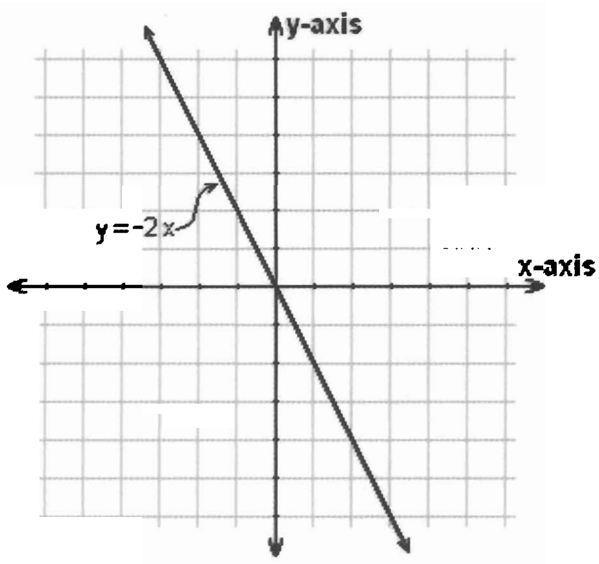
A **linear function** is a function of the form $\boxed{f(x) = mx + n}$ whose graph is a straight line. When written in this form, the **slope** or **gradient** of it is *m* and the **y-intercept** is *n*.

Unless a domain for x is otherwise stated, the domain for linear functions will be assumed to be all real numbers and so the lines in graphs of all linear functions extend infinitely in both directions.

Also in linear functions with all real number domains, the range of a linear function will cover the entire set of real numbers for y, unless the slope m = 0 and the function equals a constant. In such cases, the range is simply the constant.

Examples: graphs of $f(x) = x+2$ and $f(x) = -2x$
Domain=range= $\Re$





(See also page 122)

Slopes and y-intercepts

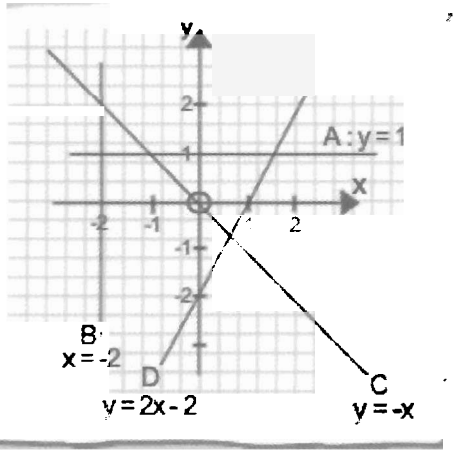
The **slope** is the ratio $\dfrac{increase \cdot in \cdot y}{increase \cdot in \cdot x}$ and the graph hits the y-axis at the **y-intercept**

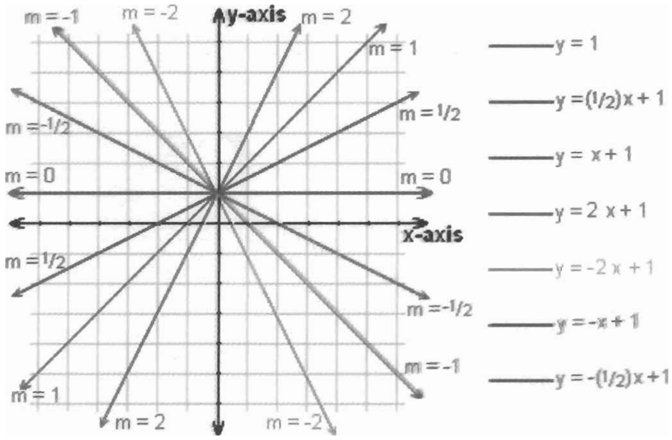
Essentially, the absolute value of the slope measures the "steepness" of the line.

If the slope is positive, then the line "rises" from left to right: increasing function; if the slope is negative, then the line "falls" from left to right: decreasing function; and if the slope is 0, then the line is horizontal: constant function.

Note: the simplest linear functions are those whose graphs pass through the origin $\boxed{f(x) = mx}$
Note: equations of the form $\boxed{f(x) = c}$ are linear functions with slope 0: **constant functions**; their graphs are straight horizontal lines, crossing through the y-axis at c.
Note: the graph of $\boxed{f(x) = 0}$ (m = 0, n = 0) is the x-axis; the the diagonals through the origin are the graphs of the linear functions $\boxed{f(x) = x}$ and $\boxed{f(x) = -x}$.
Note: the y-axis has the equation $\boxed{x = 0}$ which is not a function; because there are multiple possible values of y for the single value of x, equations of the form $\boxed{x = c}$ are not functions, but can be considered relations; the lines of such equations have no slope, although some would say the slope is infinity because the lines have infinite steepness.

Examples:





Graph A: a line parallel to the x-axis
Graph B: a line parallel to the y-axis
Graph C: a line through the origin at 45° to the x-axis
Graph D: a line with both x- and y-intercepts.

<u>Note</u>: The first thing you have to do to deal with straight-line functions is to rearrange the equation into the standard format f(x) = mx + n.

<u>Finding the intercepts of the graph of a linear function</u>

Substitute the value x=0 in the equation to calculate the y-intercept. Substitute the value y=0 in the equation to calculate the x-intercept.

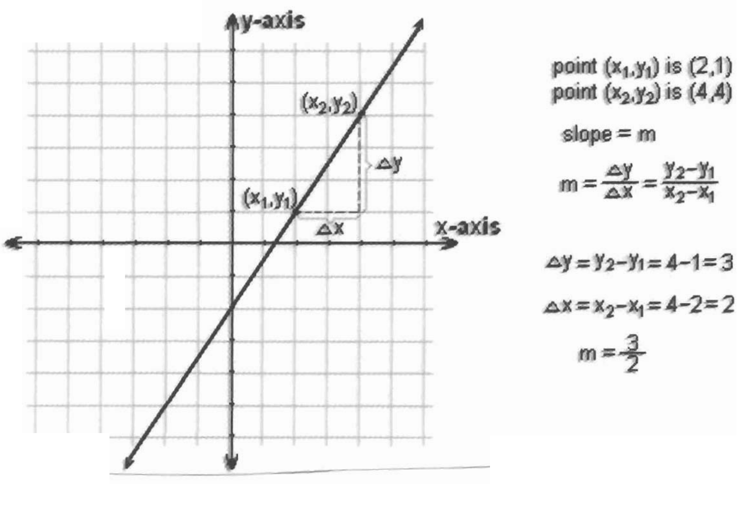
Example: suppose you are to find the intercepts of the graph given by 5x + 2y = 10
to find the x-intercept, set y = 0 and solve for x.
$5x+2\cdot 0=10 \Rightarrow 5x=10 \Rightarrow x=2$ so the x-intercept point is (2,0)
to find the y-intercept, set x = 0 and solve for y.
$5\cdot 0+2y=10 \Rightarrow 2y=10 \Rightarrow y=2$ so the y-intercept point is (0,5).

<u>Finding the equation of a linear function given its graph</u>

Once the slope and the y-intercept have been found write the equation with the standard format. The slope is calculated as follows:

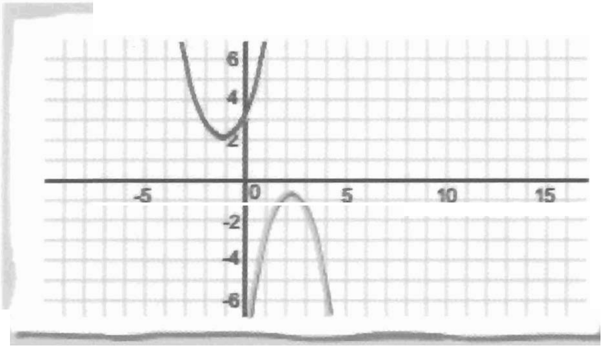
-Calculate the difference in values of the y-coordinates of two points of the graph.
-Divide it by the difference in the x-coordinates of the points ("rise divided by run").

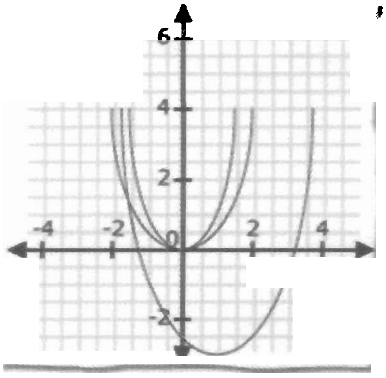
Example:



Quadratic functions

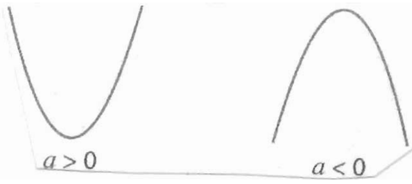
The graph of $\boxed{f(x)=ax^2+bx+c}$ is a parabola. Functions of this sort are called **quadratic functions**.





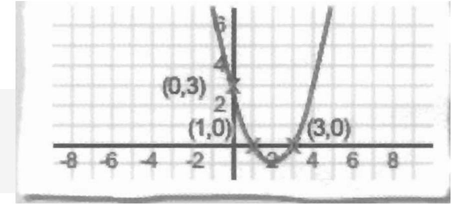
Plotting graphs of quadratic functions

The value of *a* stretches the graph and makes the function to be **convex or concave**.

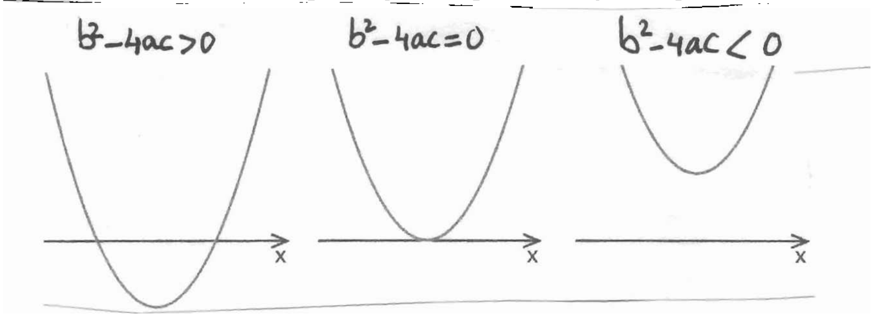


The ordinate of the **y-intercept** is found by calculating the value of f(x) when x = 0.

$f(x)=x^2-4x+3$



Some parabolas meet the x-axis. The discriminant b^2-4ac gives information about the number of **x-intercepts**. The abscissas of these points can be found by algebraic methods: factorisation of the function, completing the square or by using the formula



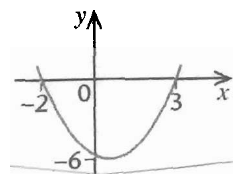
The value of $-b/(2a)$ is the abscissa of the vertex. The vertex of the curve occurs at its maximum or minimum point.

Examples:

- Sketch the graph of $y = x^2 - x - 6$.
  Find the coordinates of the lowest point on the curve.

The curve will cross the x-axis when $y = 0$. You can find these points by solving the equation $x^2 - x - 6 = 0$.

$$x^2 - x - 6 = 0 \Rightarrow (x+2)(x-3) = 0$$
$$\Rightarrow x = -2 \quad \text{or} \quad x = 3$$



The curve is symmetrical so the lowest point occurs mid-way between −2 and 3 and this is given by $(-2 + 3) \div 2 = 0.5$.

When $x = 0.5$, $y = 0.5^2 - 0.5 - 6 = -6.25$.

The lowest point on the curve is $(0.5, -6.25)$.

- Find the coordinates of the vertex of the curve $y = x^2 + 2x + 3$

You need to recognise that $x^2 + 2x + 1 = (x + 1)^2$,

then, completing the square, $x^2 + 2x + 3 = x^2 + 2x + 1 + 2 = (x + 1)^2 + 2$.

The equation of the curve can now be written as $y = (x + 1)^2 + 2$.

$(x + 1)^2$ cannot be negative so its minimum value is zero, when $x = -1$.

This means that the minimum value of y is 2 and this occurs when $x = -1$

The vertex of the curve is at $(-1, 2)$.

- The equation $5x^2 + 3x + p = 0$ has a repeated root. Find the value of p.

In this case, $a = 5$, $b = 3$ and $c = p$.
For a repeated root $b^2 - 4ac = 0$ so $9 - 20p = 0$, giving $p = 0.45$.

- Find the range of values of k for which: a) f(x)=0 has 2 distinct roots, b) f(x)=0 has 1 root, c) f(x) has no real roots, where $f(x) = 3x^2 + 2x + k$.

First of all, work out what the discriminant is: $b^2 - 4ac = 2^2 - 4 \times 3 \times k$
$= 4 - 12k$

a) Two distinct roots means:
$b^2 - 4ac > 0 \Rightarrow 4 - 12k > 0$
$\Rightarrow 4 > 12k$
$\Rightarrow k < \frac{1}{3}$

b) One root means:
$b^2 - 4ac = 0 \Rightarrow 4 - 12k = 0$
$\Rightarrow 4 = 12k$
$\Rightarrow k = \frac{1}{3}$

c) No roots means:
$b^2 - 4ac < 0 \Rightarrow 4 - 12k < 0$
$\Rightarrow 4 < 12k$
$\Rightarrow k > \frac{1}{3}$

Sketch the graphs of the following quadratic functions:

① $y=2x^2-4x+3$

② $y=8-2x-x^2$

A The first thing you need to know is whether the graph's going to be u-shaped or n-shaped (upside down). To decide, look at the coefficient of x^2.

$y=2x^2-4x+3$

The coefficient of x^2 here is positive... ...so the graph's u-shaped. +ve

$y=8-2x-x^2$

The coefficient of x^2 here is negative... ...so the graph's upside down (n-shaped). –ve

B Now find the places where the graph crosses the axes (both the y-axis and the x-axis).

(i) Put x=0 to find where it meets the y-axis.

$y=2x^2-4x+3$

$y=(2\times0^2)-(4\times0)+3$ so $y=3$

That's where it crosses the y-axis

(ii) Solve y=0 to find where it meets the x-axis.

$2x^2-4x+3=0$

$b^2-4ac=-8<0$

You could use the formula. But first check b^2-4ac to see if $y=0$ has any roots.

So it has no solutions, and doesn't cross the x-axis.

(i) Put x=0.

$y=8-2x-x^2$

$y=8-(2\times0)-0^2$ so $y=8$

(ii) Solve y=0.

$8-2x-x^2=0$

$\Rightarrow(2-x)(x+4)=0$

$\Rightarrow x=2$ or $x=-4$

This equation factorises easily...

C Finally, find the minimum or maximum.

Since $y=2(x-1)^2+1$

By completing the square

the minimum value is $y=1$, which occurs at $x=1$

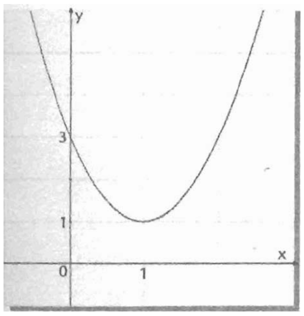
The minimum or maximum of the graph is always at $x=\frac{-b}{2a}$

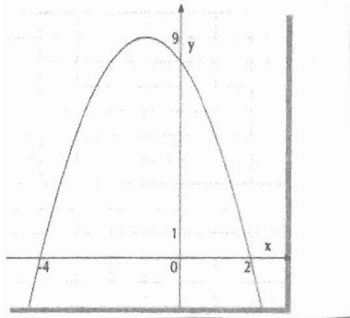
The maximum value is at $x=-1$

The maximum value is halfway between the roots — the graph's symmetrical.

So the maximum is $y=8-(2\times-1)-(-1)^2$

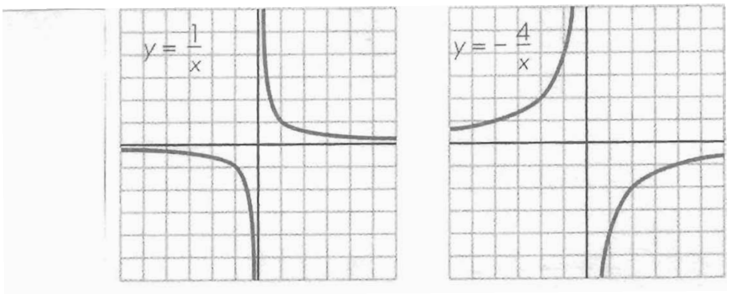
i.e. the graph has a maximum at the point (–1,9).





Inverse proportionality functions

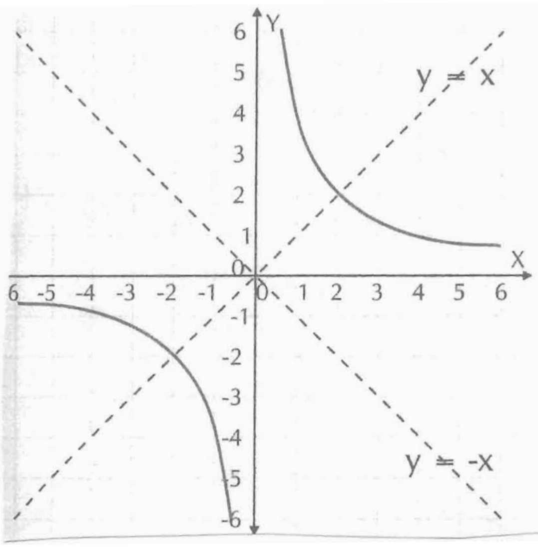
The equation of such a function is $f(x)=\frac{k}{x}$ where k is some real number.



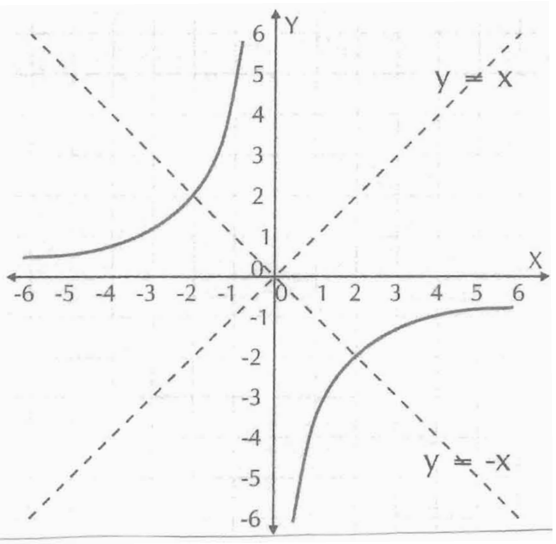
We can also consider functions like $f(x)=\frac{a}{mx+n}+b$ or $f(x)=\frac{ax+b}{mx+n}$ which are obtained by means of a translation of one of the main type.

The graph of these functions is a curve called **hyperbola.** The two halves of the graph don't touch but approximate to a pair of asymptotes; the curve is symmetrical about a pair of lines of slopes 1 and –1.

y = 4/x



y = -4/x



Plotting inverse proportionality functions

We can find the **x-intercept** and the **y-intercept** plugging 0 where necessary.

The asymptotes are on both axis when the function is of the basic type $f(x)=\frac{k}{x}$

The **vertical asymptote** is $x=\frac{-n}{m}$ (just solve mx+n=0).

The **horizontal asymptote** is y=b when $f(x)=\frac{a}{mx+n}+b$ and $y=\frac{a}{m}$ when $f(x)=\frac{ax+b}{mx+n}$

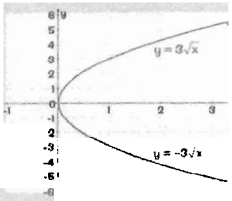
Radical functions

Let consider the functions $f(x)=k\sqrt{x}$ or more generally $f(x)=k\sqrt{mx+n}$.

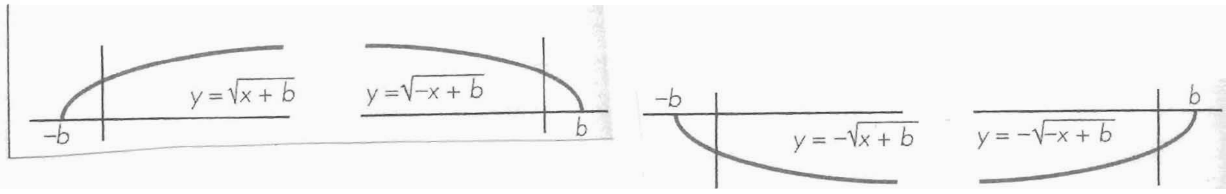
The graph of $y = k\sqrt{x}$ is a parabola on its side.

This makes sense really, because if $y = k\sqrt{x}$, then $x = \frac{1}{k^2}y^2$ —

and this is just a normal quadratic with the x and y switched round.



The vertex of the parabola is placed at $x=\frac{-n}{m}$.



EXERCISES

1) Tell the gradient of the following lines:

b) $y = 2 + 3x$

a) $y = \frac{2x+3}{5}$

c) $2y - 4x = 7$

d) $x - y = 0$

e) $4x - 3 = 5y$

f) $3y + 3x = 12$

2)

A	$y = 3x - 2$
B	$y = 7 - 2x$
C	$7y = 10x$
D	$2y = 6x + 9$
E	$x + y = 14$

(a) Which two lines are parallel?

....

(b) Which two lines have negative gradients?

....

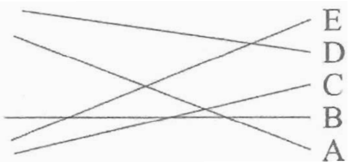
(c) Which line goes through the origin?

....

(d) Which two lines go through the point (4,10)?

....

3) Which of these lines has a positive gradient, a negative gradient or no gradient?



4) (a) Plot the line $y = 3x - 4$.
(b) Draw the line $y = 3x + 2$. Where does it lie? Explain your answer.

5) Find the equation of the straight line which passes through the points (3,1) and (-1,3)

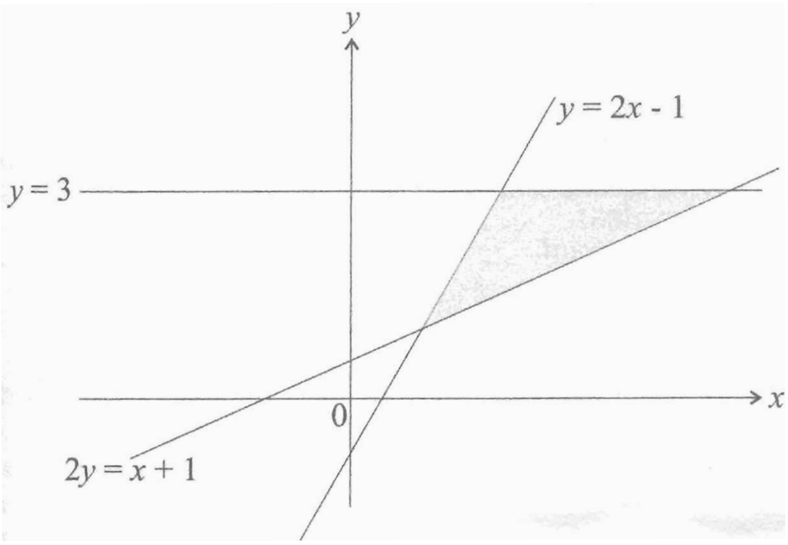
Hint: always draw a diagram to help.

6) Which is the steeper hill: 1 in 4 or 20%?

7) Without drawing, state whether the lines joining the following points form a horizontal line, a vertical line, the line y = x or the line y = -x?

(a) (1,1) to (5,5) (b) (0,4) to (-3,4) (c) (-1,3) to (-1,7) d) (4,-4) to (-3,3)

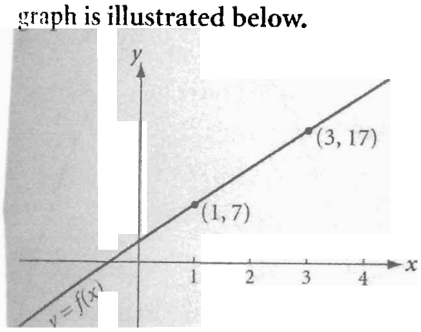
8)



Work out the coordinates of the 3 vertices of the triangle formed by the lines with equations: $y = 2x - 1$; $2y = x + 1$; $y = 3$.

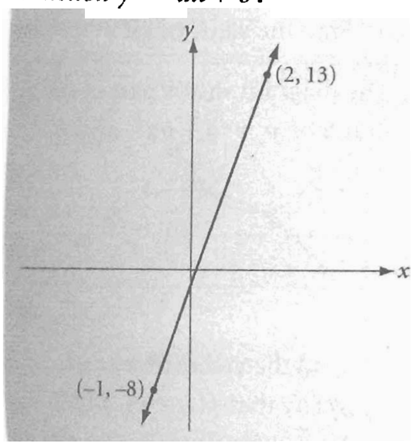
9) Plot the graph of the following, finding the x-intercepts, the y-intercept, the vertex and some more points next to it: a) $y = (x-2)^2$ b) $y = 2x^2 - 8x + 2$ c) $y = \frac{1}{3}x^2 - x + 3$ d) $y = -x^2 + 3x - 4$

10) a) $f: x \to ax + b$ is a function whose graph is illustrated below.



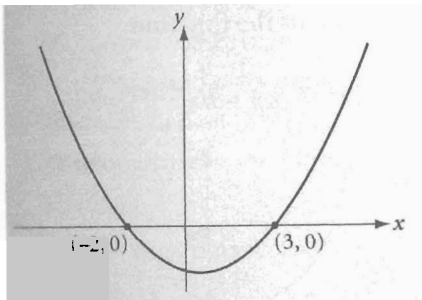
Find the values of a and b.

b) The diagram shows part of a function $y = ax + b$:



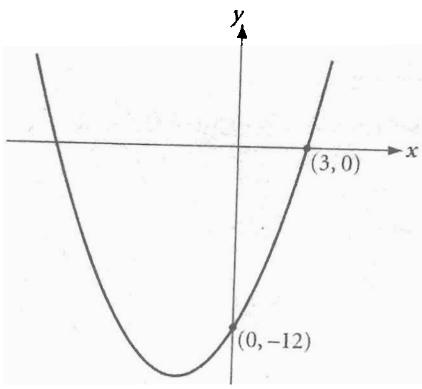
Find the values of a and b.

11) The diagram shows the graph of $f: x \to x^2 + px + q$, where $p, q \in R$.



Find the values of p and q.

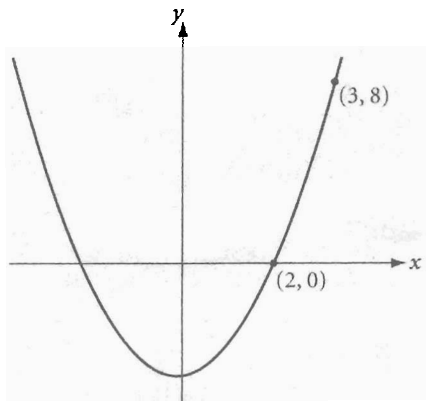
12) a) The diagram shows part of the graph of the function $f: x \to x^2 + px + q$ where $\{p, q, x\} \subset R$.



(i) Find the value of p and of q.

(ii) If $(x, 0)$ is a point on the graph (where $x \neq 3$), find the value of x.

b) The diagram shows part of the graph of the function $y = x^2 + px + q$ where $p, q \in R$.



(i) Find the values of p and q.

(ii) If $(0, n)$ is on the graph, find n.

13) Plot the graph of the following functions:

a) $y = \frac{1}{x} + 2$

b) $y = -\frac{1}{x}$

c) $y = \frac{8}{x}$

d) $y = \frac{2}{x} + 1$

14) Calculate the domain and plot the graphs of the following: $y = \frac{1}{x-4}$ $\quad y = \frac{-2}{x+1}$

$y = \frac{1}{x} + 2$ $\quad y = \frac{1}{x-1} + 1$

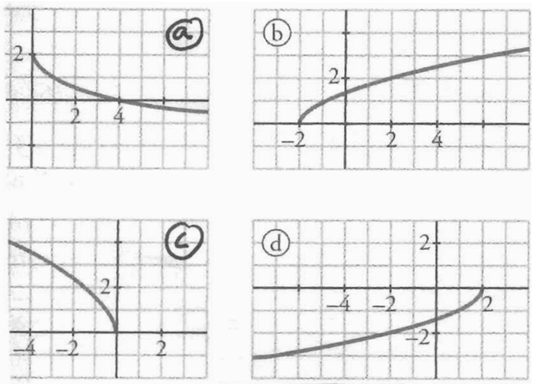
15) Match the functions with their graphs:

$y = \sqrt{x+2}$

$y = 2 - \sqrt{x}$

$y = -\sqrt{2-x}$

$y = \sqrt{-3x}$



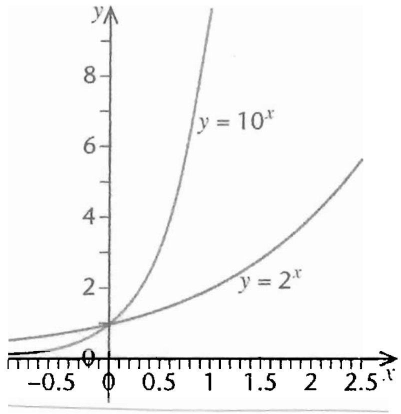
16) Calculate the domain and plot the graphs of the following:

a) $y = \sqrt{x+2}$ b) $y = \sqrt{4-x}$

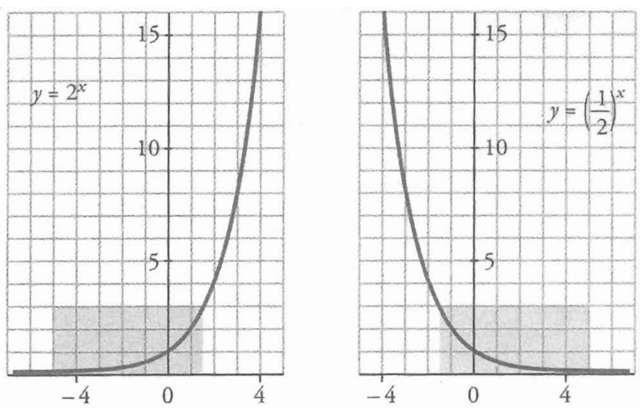
c) $y = \sqrt{2x-5}$ d) $y = 1 + \sqrt{-2x}$

Exponential functions

An **exponential function** is one where the variable is a power or exponent. For example, any function of the form $f(x) = a^x$, where a is a constant, is an exponential function.

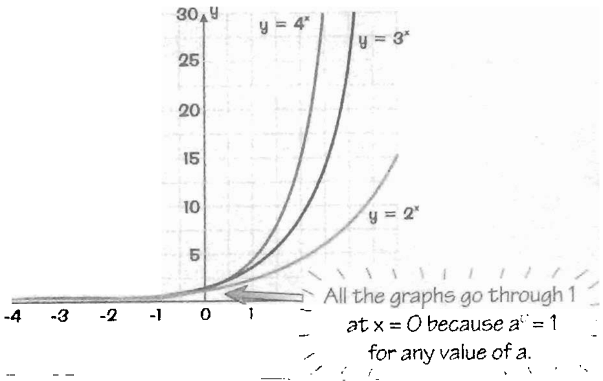


All graphs of exponential functions $f(x) = a^x$ cross the y-axis at (0,1).
The abscissas axis is a horizontal asymptote.
The domain is R and the range is $(0, +\infty)$.
The function is increasing when $a > 1$ and it is decreasing when $a < 1$



Note: Exponential functions are often used to represent patterns of growth or decay.

All the a's are greater than 1 — so y increases as x increases.
The bigger a is, the quicker the graphs increase.
The rate at which they increase gets bigger too.
As x decreases, y decreases at a smaller and smaller rate — y will approach zero, but never actually get there.



Logarithms

The **logarithm of "b" to the base "a"** is a certain number "c" such that $a^c = b$. The notation for it is $\boxed{\log_a b = c}$

Examples:

Index notation: $10^2 = 100$

log notation: $\log_{10} 100 = 2$

The base goes here but it's usually left out if it's 10.

Write down the values of the following:

a) $\log_2 8$ b) $\log_9 3$ c) $\log_5 5$

a) 8 is 2 raised to the power of 3
so $2^3 = 8$ and $\log_2 8 = 3$

b) 3 is the square root of 9, or $9^{1/2} = 3$
so $\log_9 3 = 1/2$

c) anything to the power of 1 is itself
so $\log_5 5 = 1$

Write the following using log notation:

a) $5^3 = 125$ b) $3^0 = 1$

You just need to make sure you get things in the right place.

a) 3 is the power or logarithm that 5 (the base) is raised to to get 125
so $\log_5 125 = 3$

b) you'll need to remember this one:
$\log_3 1 = 0$

For every positive number "a" $\boxed{\log_a a = 1}$ and $\boxed{\log_a 1 = 0}$

Base 10 is the most common — and the log button on your calculator gives logs to base 10.

Laws of logarithms:

$$\log_a x + \log_a y = \log_a (xy)$$
$$\log_a x - \log_a y = \log_a \left(\frac{x}{y}\right)$$
$$\log_a x^k = k \log_a x$$

- One of the main use of them is to **change of base.** It is very important because calculators find logarithms only to the **base 10** and to the **base e=2,71828...**

$$\log_a x = \frac{\log_b x}{\log_b a}$$

Example: Calculate $\log_7 4$ to 4 decimal places.

$\log_7 4 = \frac{\log_{10} 4}{\log_{10} 7} = 0.7124$. (To check: $7^{0.7124} = 4$)

- We can use the laws of logarithms to operate with them:

Examples: Write each expression in the form $\log_a n$, where n is a number.

a) $\log_a 5 + \log_a 4$ b) $\log_a 12 - \log_a 4$ c) $2 \log_a 6 - \log_a 9$

a) Use the law of logarithms

$\log_a x + \log_a y = \log_a (xy)$

You just have to multiply the numbers together:

$\log_a 5 + \log_a 4 = \log_a (5 \times 4)$

$= \log_a 20$

b) Use the law of logarithms

$\log_a x - \log_a y = \log_a (\frac{x}{y})$

Divide the numbers:

$\log_a 12 - \log_a 4 = \log_a (12 \div 4)$

$= \log_a 3$

c)

$\log_a x^k = k \log_a x$

$2 \log_a 6 = \log_a 6^2$

$\log_a 36 - \log_a 9$

$= \log_a 4$

- **We can solve equations with unknown indices.**

Examples: Use logarithms to solve the following for x, giving the answers to 4 s.f.

a) $10^x = 170$ b) $10^{3x} = 4000$ c) $7^x = 55$ d) $\log_{10} x = 2.6$ e) $2^{4x} = 80$

a) $10^x = 170$. Taking logs to base 10 of both sides gives $x = \log_{10} 170 = 2.230$.

b) $10^{3x} = 4000$. Taking logs of both sides gives $3x = \log_{10} 4000 = 3.602$, so $x = 1.201$.

c) $7^x = 55$. Take logs of both sides, and use the log rules. It doesn't matter what base you use, so why not use base 10? (Make sure you use the same base for each side though.)

$x \log_{10} 7 = \log_{10} 55$ so $x = \frac{\log_{10} 55}{\log_{10} 7} = 2.059$

d) $\log_{10} x = 2.6$. You've got to be able to go back the other way, so $x = 10^{2.6} = 398.1$

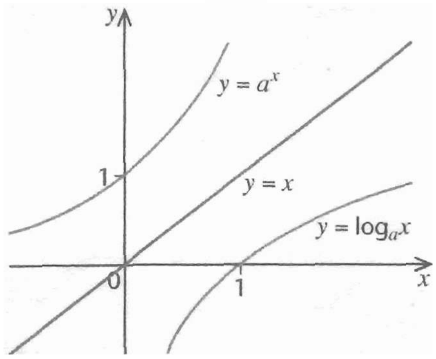
e) $2^{4x} = 80$. Take logs — again use base 10: $4x (\log_{10} 2) = \log_{10} 80$, so $x = \frac{\log_{10} 80}{4 \log_{10} 2} = 1.580$

f) $3^{1-x^2} = 1/27$. P.t $3^{1-x^2} = 3^{-3} \rightarrow 1 - x^2 = -3 \rightarrow x = 2;\ x = -2$

Logarithmic function

A **"log" function** is defined by $\boxed{f(x) = \log_a x}$, a>0.

The graph of $y = \log_a x$ is a reflection of $y = a^x$ in the line y=x.



All graphs of logarithmic functions $f(x) = \log_a x$ cross the x-axis at (1,0).
The ordinates axis is a vertical asymptote.
The domain is (0,+∞) and the range is R.
The function is increasing when a>1 and it is decreasing when a<1.

EXERCISES

17) Plot the following functions with the help of a calculator and graph paper:
a) $y=1.5^x$ b) $y=0.8^x$

18) Plot the following functions calculating a table for x and y values:
a) $y=2^{-x}$ b) $y=0.75^x$ c) $y=1.6^x$ d) $y=2^x-3$ e) $\log_3 3$

19) Evaluate the following:
(a) $\log_2 8$
(b) $\log_9 3$
(c) $2\log_4 2$

20) *Write down the values of the following*
(a) $\log_3 27$ (b) $\log_3(\frac{1}{27})$ (c) $\log_3 18 - \log_3 2$

21) Evaluate the following (calculator not allowed):
a) $log_2\ 64$ b) $log_2\ 16$
c) $log_2\ \frac{1}{4}$ d) $log_2\ \sqrt{2}$
e) $log_3\ 81$ f) $log_3\ \frac{1}{3}$
g) $log_3\ \sqrt{3}$ h) $log_4\ 16$

22) Evaluate the following (use a calculator):
a) $log_2\ 13.5$ b) $log_3\ 305$
c) $log_5\ 112$ d) $log_2\ \frac{1}{7}$
e) $log_3\ 5^7$ f) $log_4\ \sqrt{725}$
g) $log_2\ 10^6$ h) $log_3\ 10^{-4}$

23) Write each of the following as a single logarithm.
(a) $\log_a x + 3\log_a y - \frac{1}{2}\log_a z$
(b) $\log_{10} x - 1$

24) *Simplify the following*
(a) $\log 3 + 2\log 5$ (b) $\frac{1}{2}\log 36 - \log 3$ (c) $\log_b(\chi^2 - 1) - \log_b(\chi - 1)$

25) Find out the base in the following logarithms: a) $log_b\ 10\,000 = 2$ b) $log_b\ 125 = 3$
c) $log_b\ 4 = -1$ d) $log_b\ 3 = \frac{1}{2}$

26) Solve for "x" (use the definition of "logarithm"):

a) $log_2 (2x - 1) = 3$ b) $log_2 (x + 3) = -1$
c) $log\, 4x = 2$ d) $log (x - 2) = 2{,}5$
e) $log (3x + 1) = -1$ f) $log_2 (x^2 - 8) = 0$

27) Solve the equation $4^x = 100$ and give your answer to 4 s.f.

28) Solve for "x" the exponential equations below: (hint: express numbers as powers)

a) $3^{x^2-5} = 81$ b) $2^{2x-3} = 1/8$ c) $2^{x+1} = \sqrt[3]{4}$ d) $2^{x+1} = 0{,}5^{3x-2}$

29) Solve for "x":

a) $3^x + 3^{x+2} = 30$

b) $5^{x+1} + 5^x + 5^{x-1} = \dfrac{31}{5}$

c) $4^x - 5 \cdot 2^x + 4 = 0$

d) $2^{x-1} + 4^{x-3} = 5$

e) $4^x - 3 \cdot 2^{x+1} + 8 = 0$

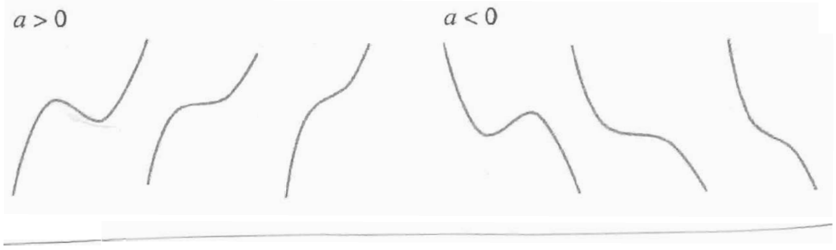
30) Given that $\log_a \chi = \log_a 4 + 3 \log_a 2$ show that $\chi = 32$

31) Plot the functions $y = 3^x$ and $y = \log_3 x$. Do the following points belong to the graph of the latter? $(243,5)$ $\left(\frac{1}{27},-3\right)$ $\left(\sqrt{3},0.5\right)$ $(-3,-1)$

Cubic functions

The equation is a third degree polynomial. $f(x) = ax^3 + bx^2 + cx + d$

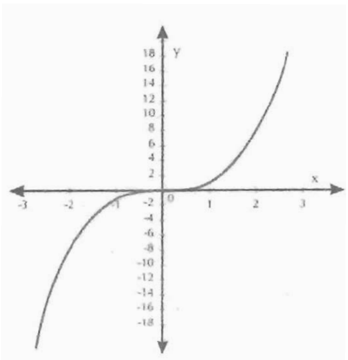
The graph of a cubic function $y = ax^3 + bx^2 + cx + d$ can take a number of forms.



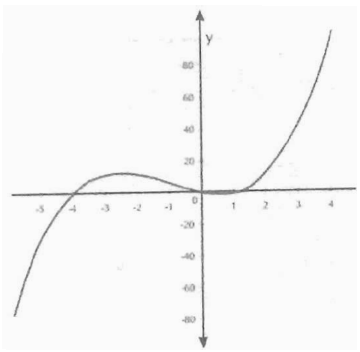
Notice that "$-x^3$ graphs" always come *down from top left* whereas the $+x^3$ ones go *up from bottom left*.

Examples:

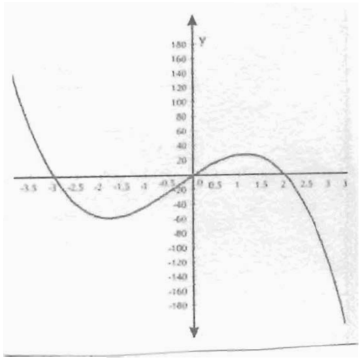
$y = x^3$



$y = x^3 + 3x^2 - 4x$

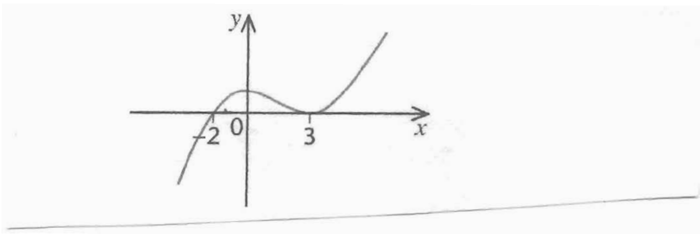


$y = -7x^3 - 7x^2 + 42x$

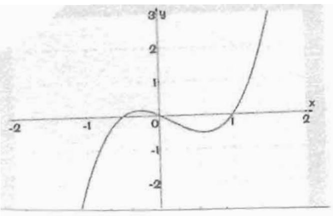


The graph of a cubic function that can be factorised as $y = (x - p)(x - q)(x - r)$ will cross the x-axis at p, q and r. If any two of p, q and r are the same then the x-axis will be a tangent to the curve at that point.

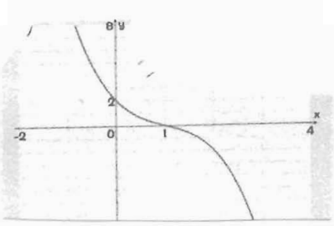
Example: the graph of $y = (x + 2)(x - 3)^2$ looks like this:



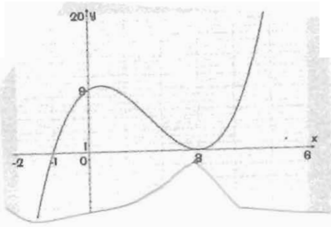
Examples:



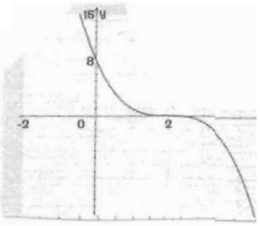
$f(x)=x(x-1)(2x+1)$



$g(x)=(1-x)(x^2-2x+2)$



$h(x)=(x-3)^2(x+1)$



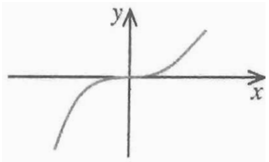
$m(x)=(2-x)^3$

Functions $y=x^n$

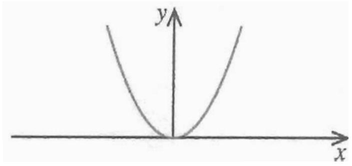
They are a special case of polynomial function when "n" is a positive integer and a special case of rational function when "n" is a negative integer.

> The graph of $y=x^n$ where n is an integer has:
> **rotational** symmetry about the **origin** when n is **odd**
> **reflective** symmetry in the y-**axis** when n is **even.**

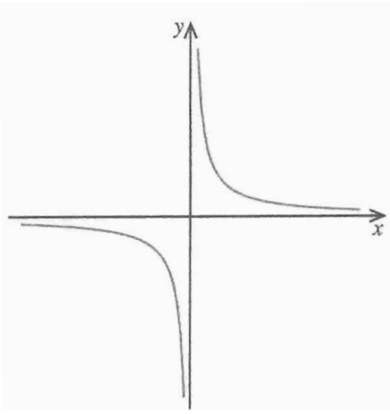
The graphs of $y=x^3$, $y=x^5$, $y=x^7$ ... look something like this:



The graphs of $y=x^2$, $y=x^4$, $y=x^6$ ... **look something like this:**

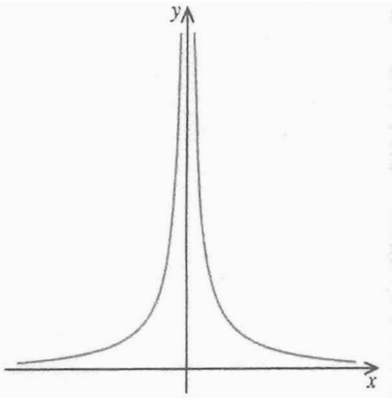


The graphs of $y=x^{-1}$, $y=x^{-3}$, $y=x^{-5}$ ... look something like this:



Both the x and y axes are asymptotes for these graphs.

The graphs of $y=x^{-2}$, $y=x^{-4}$, $y=x^{-6}$ ... look something like this:

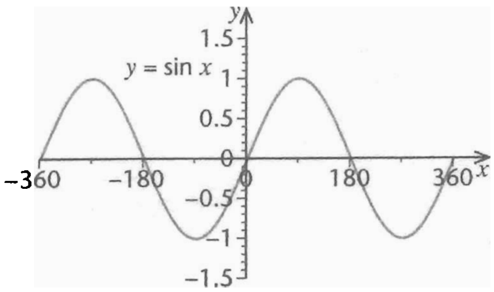


The x axis and the positive y axis are asymptotes for these graphs.

Trigonometric functions

The variable "x" stands for an angle, and "f(x)" is calculated by an expression involving some trigonometric ratios.

$y = \sin x$

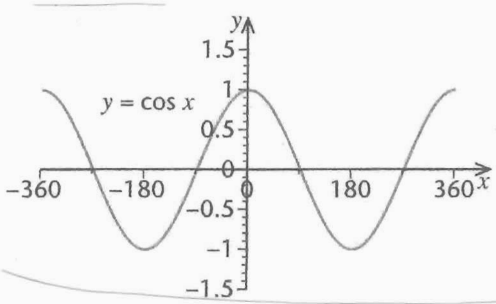


sin x is defined for any angle and always has a value between −1 and 1. It is a **periodic function** with period 360°.

The graph has **rotational symmetry** of order 2 about every point where it crosses the x-axis.

It has **line symmetry** about every vertical line passing through a vertex.

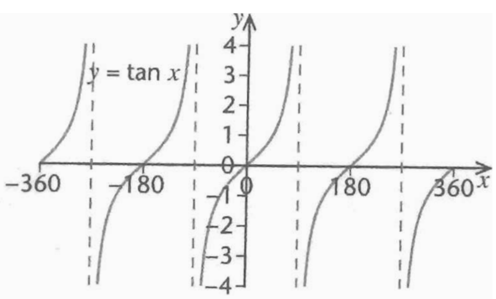
$y = \cos x$



$\cos x \equiv \sin(x + 90°)$ so the graph of $y = \cos x$ can be obtained by translating the sine graph 90° to the left.

It follows that cos x is also a periodic function with period 360° and has the corresponding symmetry properties.

$y = \tan x$

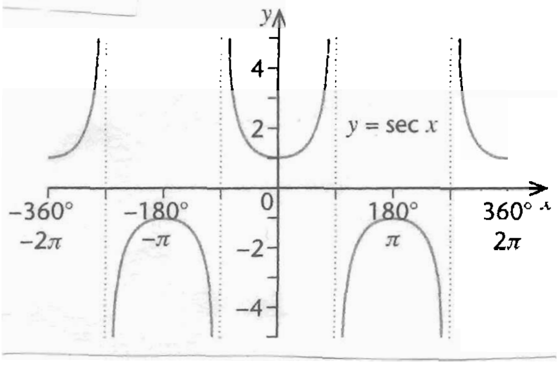


$$\tan x \equiv \frac{\sin x}{\cos x}.$$

tan x is undefined whenever $\cos x = 0$ and approaches $\pm\infty$ near these values. It is a periodic function with period 180°.

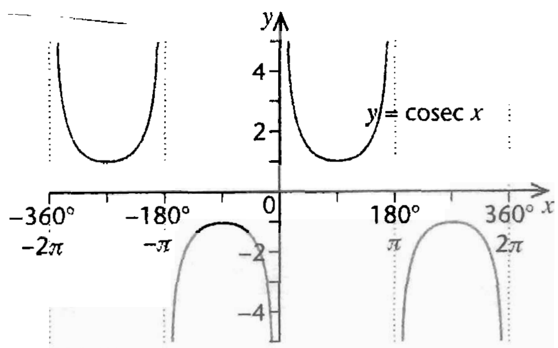
The graph has rotational symmetry of order 2 about 0°, ±90°, ±180°, ±270°, ...

$y = \sec x$



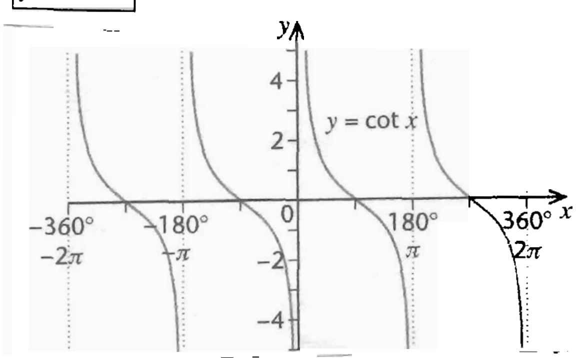
- The period of sec is 360° (2π radians) to match the period of cos.
- Notice that $\sec x$ is undefined whenever $\cos x = 0$.
- The graph is symmetrical about every vertical line passing through a vertex.
- It has rotational symmetry of order 2 about the points in the x-axis corresponding to $90° \pm 180°n$ ($\frac{\pi}{2} \pm \pi n$).

$y = \text{cosec } x$



- The period of cosec is 360° (2π radians) to match the period of sin.
- Notice that $\text{cosec } x$ is undefined whenever $\sin x = 0$.
- The graph is symmetrical about every vertical line passing through a vertex.
- It has rotational symmetry of order 2 about the points on the x-axis corresponding to $0°, \pm 180°, \pm 360°, \ldots$, $(0, \pm\pi, \pm 2\pi, \ldots)$.

$y = \cot x$



- The period of cot is 180° to match the period of tan.
- Notice that $\cot x$ is undefined whenever $\sin x = 0$.
- The graph has rotational symmetry of order 2 about the points on the x-axis corresponding to $0°, \pm 90°, \pm 180°, \ldots$, $(0, \pm\frac{\pi}{2}, \pm\pi, \ldots)$.

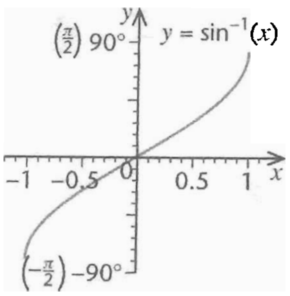
Inverse trigonometric functions

The variable "x" stands for a real number, and "f(x)" is an angle calculated by an expression involving the inverse of sine, cosine, etc.

The sine, cosine and tangent functions are all many–one and so do not have inverses on their full domains. However, it is possible to restrict their domains so that each one has an inverse. The graphs of these inverse functions are given below.

Note: When you use the functions $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$ on your calculator, the value given is called the **principal value** (PV).

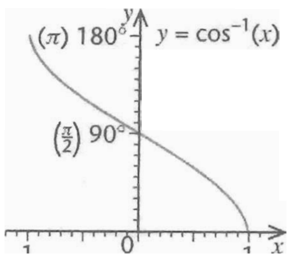
$\boxed{y = \arcsin x}$



$\sin^{-1} x$ means:
'the angle whose sine is x'

$f(x) = \sin x$, $-90° \leqslant x \leqslant 90°$ $\Rightarrow f^{-1}(x) = \sin^{-1} x$, $-1 \leqslant x \leqslant 1$.

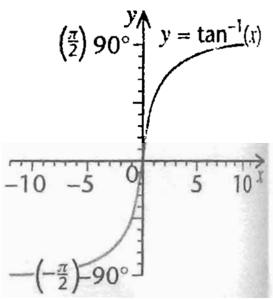
$\boxed{y = \arccos x}$



$\cos^{-1} x$ means:
'the angle whose cosine is x'

$f(x) = \cos x$, $0° \leqslant x \leqslant 180°$ $\Rightarrow f^{-1}(x) = \cos^{-1} x$, $-1 \leqslant x \leqslant 1$

$\boxed{y = \arctan x}$



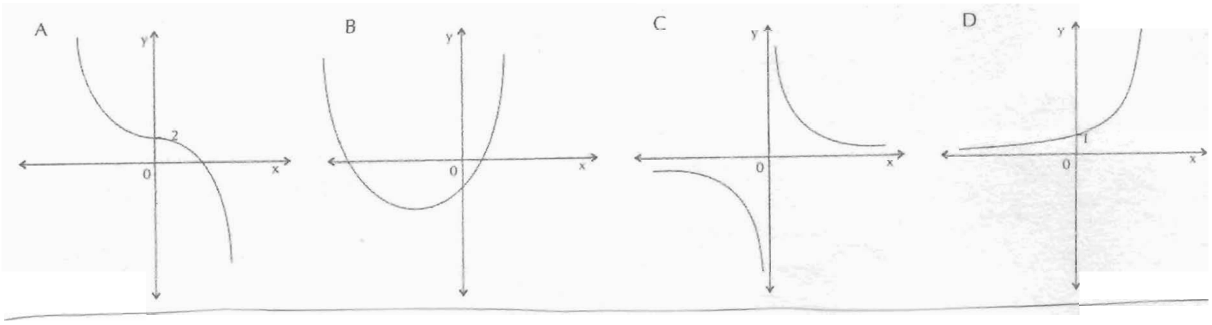
$\tan^{-1} x$ means:
'the angle whose tange is x'

$f(x) = \tan x$, $-90° < x < 90°$ $\Rightarrow f^{-1}(x) = \tan^{-1} x$, $x \in$

EXERCISES

32) Match the graphs with their equations.

(a) $y = 2^x$ (b) $y = 2x^2 + 5x - 3$ (c) $y + x^3 = 2$ (d) $xy = 1/2$



33) Plot the following with the help of a calculator and graph paper:

$y = \sin x$, $x \in \mathbb{R}$
$y = \cos x$, $x \in \mathbb{R}$
$y = \tan x$, $x \in \mathbb{R} - \{90° + k180°, k \in Z\}$
$y = \text{cosec } x$, $x \in \mathbb{R} - \{k180°, k \in Z\}$
$y = \sec x$, $x \in \mathbb{R} - \{90° + k180°, k \in Z\}$
$y = \cot x$, $x \in \mathbb{R} - \{k180°, k \in Z\}$
$y = \arcsin x$, $x \in (-1,1)$, $y \in (-90°, 90°)$
$y = \arccos x$, $x \in (-1,1)$, $y \in (0°, 180°)$
$y = \arctan x$, $x \in \mathbb{R}$, $y \in (-90°, 90°)$
$y = \text{arccosec } x$, $x \in \mathbb{R} - (-1,1)$, $y \in (-90°, 90°)$
$y = arc\sec x$, $x \in \mathbb{R} - (-1,1)$, $y \in (0°, 180°)$
$y = arc\cot x$, $x \in \mathbb{R}$, $y \in (0°, 180°)$

34) Sketch the graph of $y = 2\sin(x + 30°) + 1$ for $0° \leqslant x \leqslant 360°$

It is helpful to think about building the transformations in stages. Use a computer.

Transformations

Basic function	Translate the curve 30° to the left.	Now apply a one-way stretch with scale factor 2 parallel to the y-axis	Finally translate the curve 1 unit up.
$y = \sin x$	$y = \sin(x + 30)$	$y = 2\sin(x + 30)$	$y = 2\sin(x + 30°) + 1$