# MATHEMATICS

# YEAR 4

## *THIRD TERM*

## Cartesian product and correspondences

The **Cartesian product** of two sets A and B (also called the **product set, set direct product,** or **cross product**) is defined to be the set of all ordered pairs *(a,b)* where $a \in A$ and $b \in B$. It is denoted $A \times B$.

$$A \times B = \{(a,b) \mid a \in A, b \in B\}$$

Some relations map some points in a set *A* to one or several points in a set *B*. These relations can be saw as a subset of $A \times B$ and are called **correspondences.**

Example:
$A=\{1,2,3\}$ $B=\{a,b,c,d\}$
$A \times B.=\{(1,a),(1,b),(1,c),(1,d),(2,a),(2,b),(2,c),(2,d),(3,a),(3,b),(3,c),(3,d)\}$
a correspondence $\{(2,d),(3,c)\}$

Types of correspondences:
- **left-total**: for all $a \in A$ there exists a $b \in B$ such that *a* is mapped to *b*
- **right-total** or surjective: for all $b \in B$ there exists an $a \in A$ such that *a* is mapped to *b*
- **function**: for all $a \in A$ there exists a unique element $b \in B$ such that *a* is mapped to *b*

Note: a left-total correspondence where at least one point in set *A* is map to several points in *B* is called a **multivalued function** (or **multiple-valued function**).

## Functions

A **function** (or **map** or **mapping**) from *A* to *B* is a relation that associates every element in *A* to a unique element in *B*. It is an object *f* such that for every $a \in A$, there is a unique element $f(a) \in B$.
The set *A* is called the **domain** (the set of "imputs")
The set *B* is called the **codomain** (the set of allowable "outputs")
The **range** of the function *f* is the subset of elements of the codomain which correspond to some element in the domain.

Example: domain $A=\{1,2,3\}$ codomain $B=\{a,b,c,d\}$
a function $f=\{(1,d),(2,d),(3,c)\}$
$f(1)=d$ $f(2)=d$ $f(3)=c$
range $\{d,c\}$

Note: Generally speaking, the symbol *f* refers to the function itself, while *f(x)* refers to the value taken by the function when evaluated at a point.

## Functions from a number set to a number set

We can mainly distinguish between functions over the reals and functions over the naturals. The latter are known as **sequences.**

## Functions over the reals

When a function is **defined with words** its description is usually vague and probably inaccurate.
Examples: the image of "x" under *f* is the difference between its cube and its product by nine
a function *g* maps every real number to its square

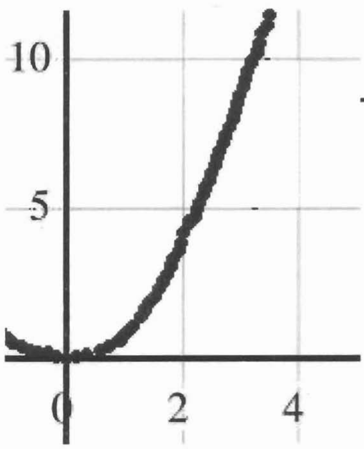
A function can be specified by **tabulating** the arguments x and their corresponding values *f*(x)
If the domain is finite, a function *f* may be perfectly defined this way.
Examples:

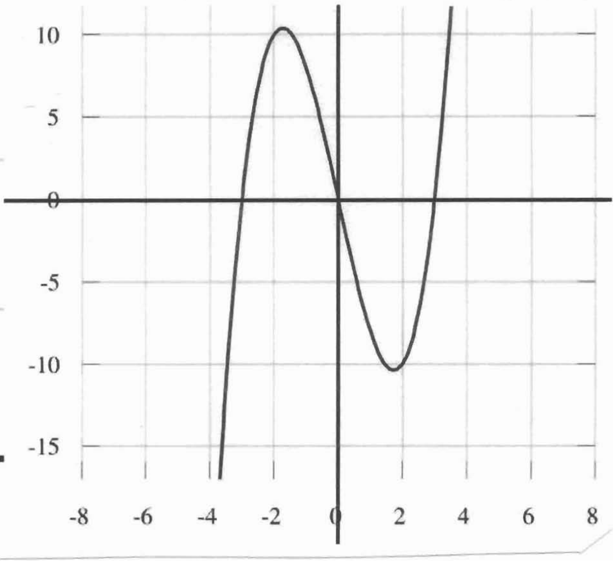
| x | –4 | –3 | –2 | –1 | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|---|---|---|---|
| y | –28 | 0 | 10 | 8 | 0 | –8 | –10 | 0 | 28 |

| x | –4 | –3 | –2 | –1 | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|---|---|---|---|
| g(x) | 16 | 9 | 4 | 1 | 0 | 1 | 4 | 9 | 16 |

A function can be specified by means of a graph.
The **graph of a function** *f* is the collection of all ordered pairs (x,*f*(x)).
In particular, **graph** means the graphical representation of this collection in a Cartesian coordinates system. Graphing on a Cartesian plane is sometimes referred to as **curve sketching**.

Examples




More commonly, a function is defined by a **formula** or an **algorithm** (that is, a recipe that tells how to compute the value of *f*(x) given any x in the domain).
Notations commonly used to represent functions from a subset of R to R:
- The notation $f: x \mapsto f(x)$ specifies that *f* is a function acting upon a single number *x* and returning a value *f(x)*.
- We can use variable *y* to represent the value associated to *x* (mainly when a name for the function is not needed)
- The notation $f: A \mapsto B$, where $f(x) = ...$ is used to explicitly specify the domain of the function.

Examples:

| | | | |
|---|---|---|---|
| $f: x \mapsto x^3 - 9x$ | $y = x^3 - 9x$ | $f(x) = x^3 - 9x$ | $f: \mathrm{R} \mapsto \mathrm{R}$ where $f(x) = x^3 - 9x$ |
| $g: x \mapsto x^2$ | $y = x^2$ | $g(x) = x^2$ | $g: \mathrm{Z} \mapsto \mathrm{R}$ where $g(x) = x^2$ |

## Some vocabulary of functions and graphs

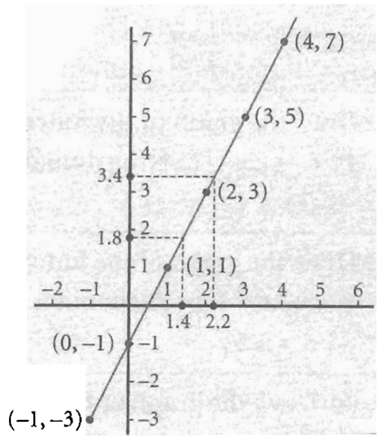
If you write a function *f* in the form y = *f*(x), y is considered **dependent** on x and x is said to be the **independent variable.**
A specific input value to a function is called an **argument of the function.**
For each argument x, the corresponding unique y in the codomain is called **the function value at x,** or **the image of x under *f*.**
The image of x may be written as ***f*(x)** or as **y**.

The **graph** of a function *f* from real numbers over real numbers is the set of all ordered pairs (x, *f*(x)), for all x in the domain.
These ordered pairs are the Cartesian coordinates of points (x is the abscisa and *f*(x) is the ordinate)

**Example**

$f: x \rightarrow 2x-1$ is a function. This function can be written in various other ways:
$y = 2x-1$; $f(x) = 2x-1$; $x \rightarrow 2x-1$

| x | y |
|---|---|
| -1 | -3 |
| 0 | -1 |
| 1 | 1 |
| 2 | 3 |
| 3 | 5 |
| 4 | 7 |




If we plot these points, we see that they form a straight line. By joining these points we form the graph of this function.

'If the input is 2.2, find the output'. Draw a dotted line from 2.2 on the *x*-axis, to the graph and then to the *y*-axis.

the output is 3.4

'If the output is 1.8, find the input'. Draw a dotted line from 1.8 on the *y*-axis to the graph and then to the *x*-axis.

the input is 1.8

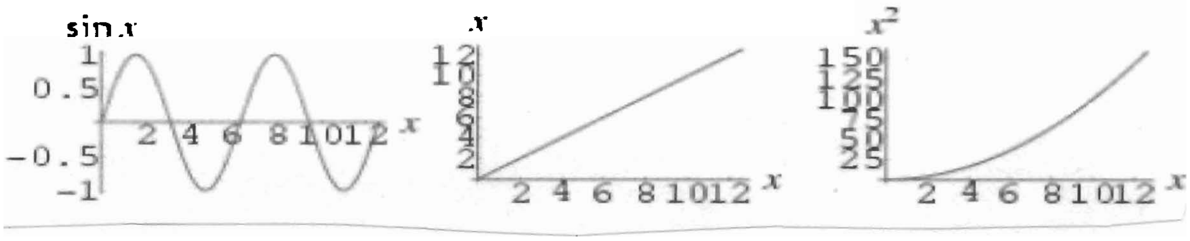
If A is any subset of the domain, then ***f*(A)** is the subset of the range consisting of all images of elements of A. We say the *f*(A) is **the image of A under *f*.**
If B is any subset of the codomain, then the subset $f^{-1}$(B) = {x in X | f(x) is in B} is the **preimage** (or **inverse image**) **of B under *f*.**

Note: the range of *f* is the image of its domain.

Some functions over the reals map many numbers to the same number: **many-to-one** functions
In other functions over the reals different numbers of the range correspond to different numbers of the domain: **one-to-one** functions

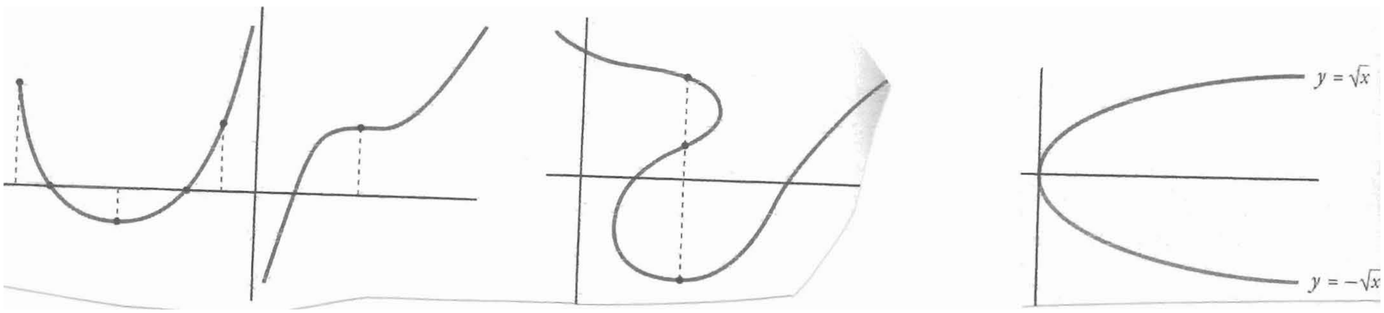
Examples:

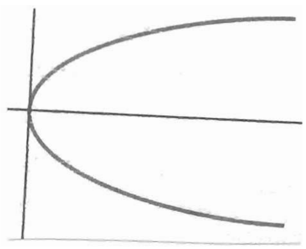
f(x)=sinx (many-to-one) f(x)=x (one-to-one) f(x)=$x^2$ (two-to-one but for x=0)




Note: some graphs of relations reveal that the relation is not a function; that happens when more than one element of the codomain are correspondent to an element of the domain.

is a function are not functions can be split into two functions

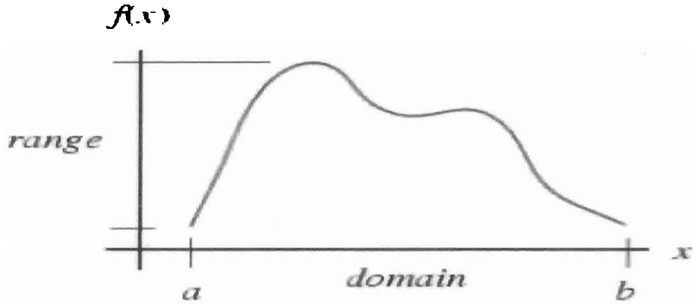


## Characteristics of a real function over real numbers

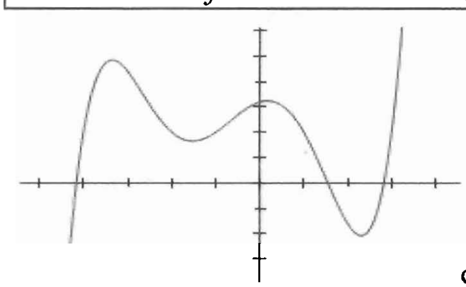
**Let $f$ be a function over real numbers given by $y = f(x)$**

| | |
|---|---|
| **Domain** of $f$ | Domain($f$) = $\{x \in \mathbb{R} \mid \exists f(x)\}$ |
| **Range** of $f$ | Range($f$) = $\{y \in \mathbb{R} \mid \exists x \in \mathbb{R}, y=f(x)\}$ |

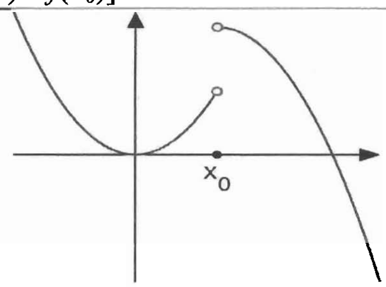



**Intercept points;**
if $0 \in$ Domain($f$) then the **vertical intercept** or the **y-intercept point** is $(0, f(0))$
if $x_1, x_2, \ldots x_n \in$ Range($f$) and satisfy $0 = f(x)$ then the **horizontal intercepts** or the **x-intercept points** are $(x_1, 0), (x_2, 0), \ldots (x_n, 0)$

**Continuity;** $f$ is **continuous** if its graph can be drawn without lifting the pencil from the paper
$f$ is **continuous** if small changes in the input result in small changes in the output.
$f$ is **continuous at $x_0$** if close to $x_0$ arguments have close to $f(x_0)$ images
$f$ is **continuous at $x_0$** if $[x \to x_0 \Rightarrow f(x) \to f(x_0)]$

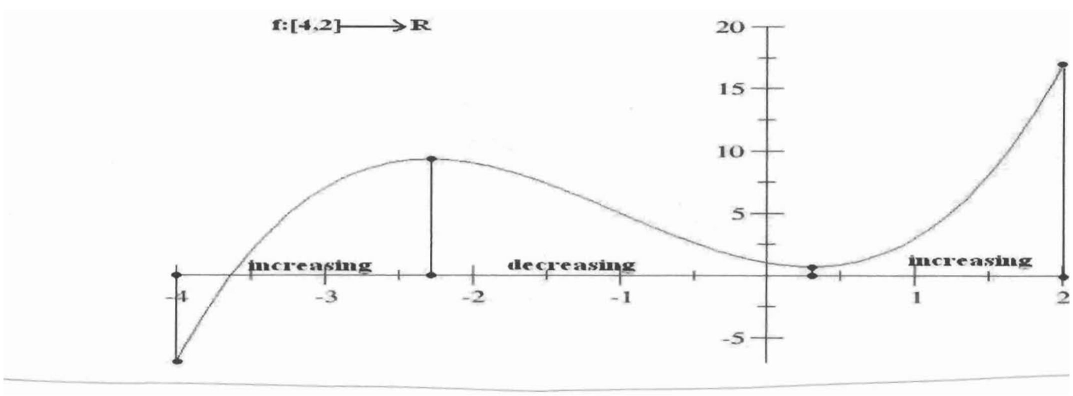


continuous




discontinuous at $x_0$

**Monotonicity;**
$f$ is **increasing on the interval I** if $\forall x, x' \in I\ [x < x' \Rightarrow f(x) < f(x')]$
$f$ is **non increasing on the interval I** if $\forall x, x' \in I\ [x < x' \Rightarrow f(x) \geq f(x')]$
$f$ is **decreasing on the interval I** if $\forall x, x' \in I\ [x < x' \Rightarrow f(x) > f(x')]$
$f$ is **non decreasing on the interval I** if $\forall x, x' \in I\ [x < x' \Rightarrow f(x) \leq f(x')]$



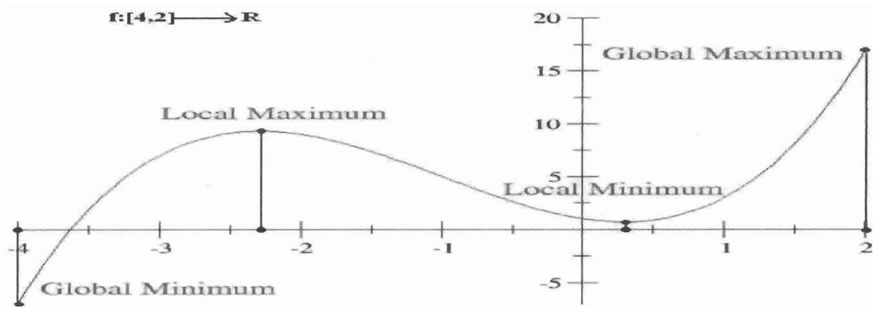
**Extrema (turning points);**

*f* has a **local** or **relative maximum at $x_0$** if $\exists$ (a,b)$\ni x_0$ | $\forall x\in$(a,b) $f(x)\leq f(x_0)$

*f* has a **local** or **relative minimum at $x_0$** if $\exists$ (a,b)$\ni x_0$ | $\forall x\in$(a,b) $f(x)\geq f(x_0)$

*f* has a **global** or **absolute maximum at $x_0$** se $\forall x\in$Domain(*f*) $f(x)\leq f(x_0)$

*f* has a **global** or **absolute minimum at $x_0$** se $\forall x\in$Domain(*f*) $f(x)\geq f(x_0)$

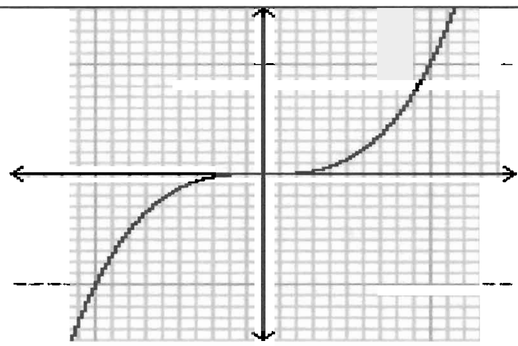



**Curvature;** *f* is **convex on the interval I** if $\forall x_1,x_2\in I\left[x\in(x_1,x_2)\Rightarrow\frac{f(x)-f(x_1)}{x-x_1}<\frac{f(x_2)-f(x_1)}{x_2-x_1}\right]$

*f* is **convex on the interval I** if $\forall x_1,x_2\in I$ and $\forall t\in[0,1]$ $f(ta+(1-t)b)\leq tf(a)+(1-t)f(b)$

*f* is **concave on the interval I** if $\forall x_1,x_2\in I\left[x\in(x_1,x_2)\Rightarrow\frac{f(x)-f(x_1)}{x-x_1}>\frac{f(x_2)-f(x_1)}{x_2-x_1}\right]$

*f* is **concave on the interval I** if $\forall x_1,x_2\in I$ and $\forall t\in[0,1]$ $f(ta+(1-t)b)\geq tf(a)+(1-t)f(b)$

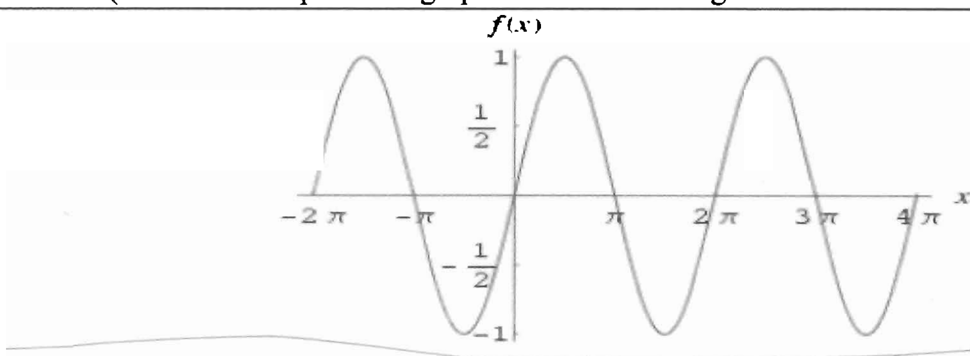


**Inflection points;** the points $x_0$ at which *f* changes its curvature are called **inflection points**

**Periods;**

*f* is a **periodic function with period T** if $\forall x\in$Domain(*f*) [x+T$\in$Domain(*f*) and *f*(x)=*f*(x+T)]

(The entire shape of the graph can be seen in a given section T units long).



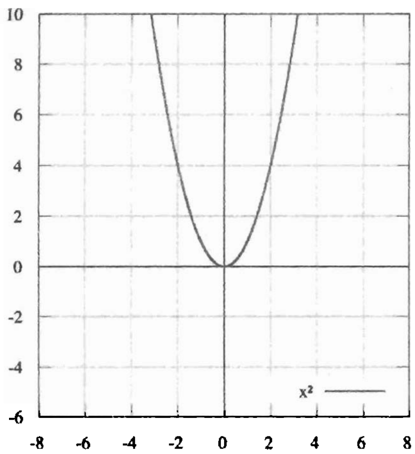
**Rate of change;**

if a,b∈Domain($f$) the **average rate of change** of $f$ **between a and b** is $\frac{\nabla y}{\nabla x}=\frac{f(b)-f(a)}{b-a}$

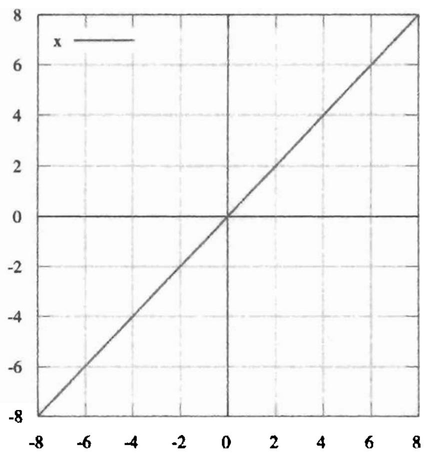
Increasing functions have positive rates of change
Decreasing functions have negative rates of change

**Symmetry;**

$f$ is **even** if $f(x)=f(-x)$ $\forall x,-x\in$Domain($f$) (the graph has a line of symmetry in the y axis)
$f$ is **odd** if $f(x)=-f(-x)$ $\forall x,-x\in$Domain($f$) (order-2 rotational symmetry about the origin)



even function



odd function

**End behaviour** and **asymptotes;**

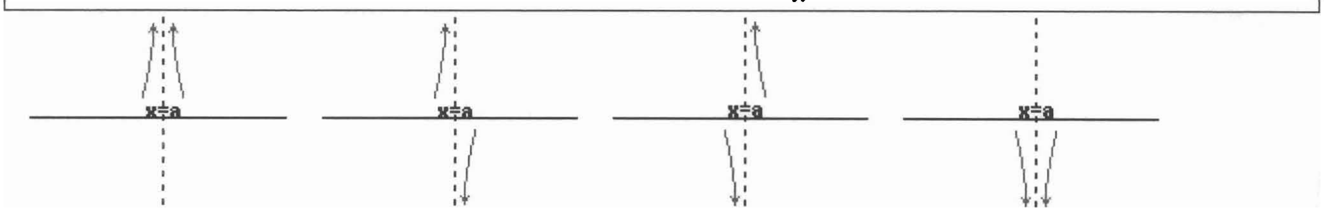
the **end behaviour** can be $x\to+\infty \Rightarrow f(x)\to+\infty$
or else $x\to+\infty \Rightarrow f(x)\to-\infty$
or else $x\to-\infty \Rightarrow f(x)\to+\infty$
or else $x\to-\infty \Rightarrow f(x)\to-\infty$

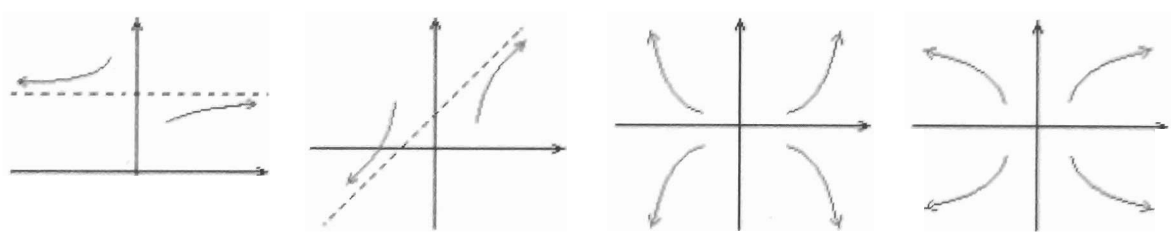
the line x=a is a **vertical asymptote** for $f$ if $[x\to a\Rightarrow f(x)\to\pm\infty]$
the line y=b is an **horizontal asymptote** for $f$ if $[x\to\infty\Rightarrow f(x)\to b]$
the line y = mx+n is a **slant** or **oblique asymptote** for $f$ if $[x\to\infty\Rightarrow f(x)-mx\to n]$

$f$ has a **parabolic asymptote** when $x\to\infty \Rightarrow f(x)\to\infty$ and $x\to\infty\Rightarrow\frac{f(x)}{x}\to\infty$

or else when $x\to\infty \Rightarrow f(x)\to\infty$ and $x\to\infty\Rightarrow\frac{f(x)}{x}\to 0$

or else when $x\to\infty \Rightarrow f(x)\to\infty$ and $x\to\infty\Rightarrow\frac{f(x)}{x}\to m\neq 0$ and $x\to\infty \Rightarrow f(x)-mx\to\infty$



A straight line that a graph approaches ever more closely without actually touching it is called an asymptote.

## Examples on calculating domains

- For example, the function $f(x)=\dfrac{x+2}{x-3}$ cannot have 3 in its domain since division by zero is undefined.

  So Domain($f$)=$\mathbb{R}-\{3\}$

- Let us calculate the domain of $y=\dfrac{1}{x^2-2x-8}$

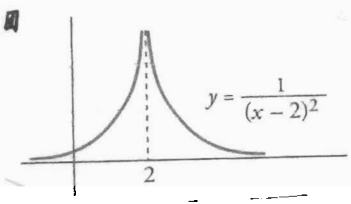
  $$x^2-2x-8=0 \rightarrow x=\frac{2\pm\sqrt{4+32}}{2}=\frac{2\pm 6}{2}=\left\langle\begin{matrix}4\\-2\end{matrix}\right.$$

  The denominator is zero for values x = –2, x = 4 so Domain= $\mathbb{R}-\{4,-2\}$

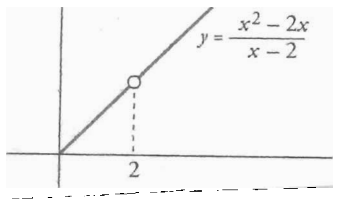
- Let us calculate the domain of $y=\sqrt{x+5}$

  The rooted must be non-negative: $x+5\geq 0 \rightarrow x\geq -5$
  So Domain=$[-5, +\infty)$

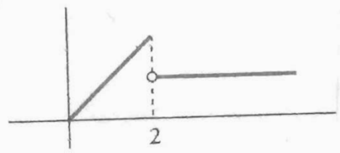
## Examples on spotting discontinuities




The function is discontinuous at x=2
This function has a vertical asymptote x=2. The function grows without an end as x approaches 2.

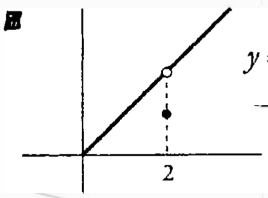



The function is discotinuous at x=2
This function is not defined at x=2.




$$y=\begin{cases} x & \text{for } x\leq 2\\ 1 & \text{for } x>2\end{cases}$$

The function is discontinuous at x=2
A piece-wise function: the function is like y=2 but for x=2, where the value of the function is y=1

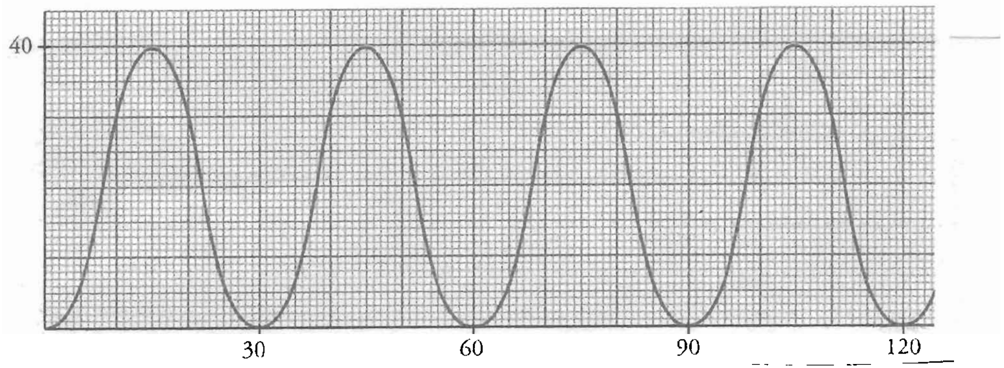



$$y=\begin{cases} x & \text{for } x\neq 2\\ 1 & \text{for } x=2\end{cases}$$

The function is discontinuous at x=2
A piece-wise function: for x ≤ 2 the function is like y=x. For x >2 the function is y=1

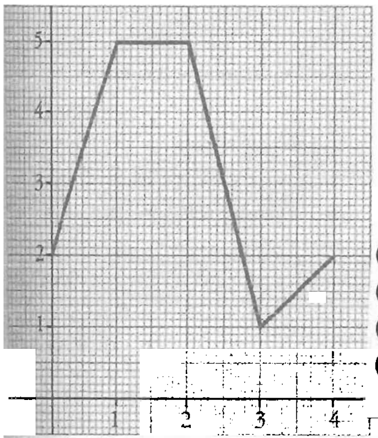
## Examples on reading graphs of periodic functions

- The change of height (m) of a point in a rotating wheel along the time (s) is represented by the following graph:




It takes 30 seconds to rotate 360°, so the period of the function is T=30

In the following graph appears the beginning of a periodic function (T=4). Let us find the images of the following arguments: x=9, x=7, x=418.5 and x=1603.5

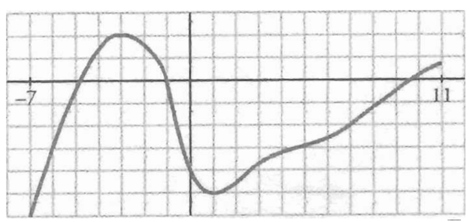



$f(9)=f(1)=5$
$f(7)=f(3)=1$
$f(418.5)=f(2.5)=3$
$f(1603.5)=f(3.5)=1.5$

(because $9 = 2\cdot 4 + 1$ the pattern is repeated "twice and a bit")
(because $7 = 4 + 3$ the pattern is repeated "once and a bit")
(because $418.5 = 104\ \ 4 + 2.5$ the pattern is repeated "104 times and a bit")
(because $9 = 400 \cdot 4 + 3.5$ the pattern is repeated "400 times and a bit")

## Examples on studying monotonicity and extrema

- Say where the function is increasing and where it is decreasing; find the local and global extrema.



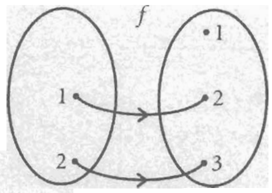

Domain=[–7, 11]
The function is increasing on $[-7,-3)\cup(1,11]$
The function is decreasing on $(-3,1)$
There is a relative maximum at x = –3 and its value is 2.
There is a relative minimum at x = 1 and its value is –5
The global maximum is reached at x = –3
The global minimum is reached at x = –7 and its value is –6

## EXERCISES

1) $f: x \to 4x - 1$. The domain of $f$ is $\{1, 2, 3, 4\}$. Find the range.

2) $g: x \to 2x^2 + 1$. The domain of $g$ is $\{0, 1, 2\}$. Find the range.

3) From the Venn diagram on the right, list the elements of:
(i) the domain of $f$.
(ii) the codomain of $f$.
(iii) the range of $f$.




4) Give an example of cartesian product, another example of correspondence and another example of function.

5) Give an example of function over ℝ and specify it by means of a formula, a table, a graph and describing it with words.

6) Calculate the images under $f(x) = 2x^3 - x + 4$ and under $g(x) = \dfrac{3x^2 - 4}{5}$ of the desired arguments:

$f(1)$ $f(-5)$ $g(2)$ $g(-1)$ $f(2)$ $f(-3)$ $g(0)$ $g\left(\frac{1}{2}\right)$

7) Calculate the domains of the following functions:

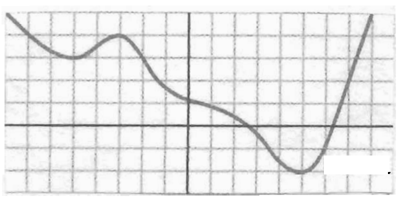
$y = \dfrac{1}{x^2 + 2x - 8}$ $y = \sqrt{x - 5}$ $y = \sqrt{x^2 - 2x - 8}$ $y = \sqrt{x + 5}$

$y = \dfrac{1}{x^2 - 2x - 8}$ $y = \sqrt{x^2 + 2x - 8}$ $y = \dfrac{1}{\sqrt{x + 5}}$

8) Calculate the x-intercept and the y-intercept points of the graphs of the following function:

$f(x) = -3x + 42$ $g(x) = \dfrac{4x + 4}{5x + 2}$ $h(x) = 3x^2 + x - 2$ $k(x) = 3 - \sqrt{25 - 2x}$

9) On what intervals is the function increasing?
On what intervals is it decreasing?
At which points does it have maxima and minima?

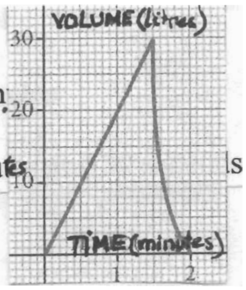


10) Calculate the rate of change of $f(x) = 2x^3 - x + 4$
a) between 2 and 6 b) between –3 and 1 c) between 0 and 10

Calculate the rate of change of $g(x) = \dfrac{3x^2 - 4}{5}$
a) between –2 and 0 b) between 3 and 8 c) between 1 and 4

11) Give an example of concave function, an example of convex function and a function with an inflection point.

12) The cistern of a public toilet empties each two minutes as shown in the graph:
a) Complete the graph corresponding to the content of water during 10 min
b) How much water is there in the cistern at the following moments:
After 17 min ______ After 40 minutes and 30 seconds ______ After 1 hour, 9 minutes and 30 seconds

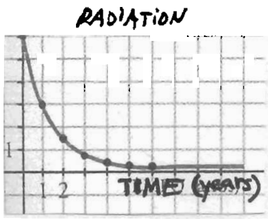



13) Plot the graph of the sine function and say what its period is.

14) Give an example of an even function and another example of odd function.

15) Give an example of a function with an horizontal asymptote, another example with a vertical asymptote and a function with and end behaviour of growing to $+\infty$ when $x \to \pm\infty$

16) The amount of radiation of a substance decreases by a half in a year. The graph shows the amount of radiation of an object along the time. What value does the radiation tend to as time passes by?




17) Draw a graph showing how the temperature of a piece of ice changes. The temperature was –10°C initially, and after 0.5 h it was 0°C; after 2 more hours the ice was finally melt. The environment temperature was 20°C.

18) What does the area of a circle tend to as the radius grows?

19) Plot the graphs of

$$f(x)=\begin{cases} \frac{x}{x+1} & \text{if } x<-1 \\ 3 & \text{if } -1\le x<4 \\ (x-4)^2+3 & \text{if } 4\le x \end{cases}$$

$$g(x)=\begin{cases} -x^2-4x-1 & \text{for } x<-3 \\ x+5 & \text{for } -3\le x<1 \\ \frac{1}{x-5}+5 & \text{for } 1\le x \end{cases}$$

## Operations with functions

Just like you can add, subtract, multiply, or divide numbers, you can do those same operations with functions. Suppose you have two functions $f(x)$ and $g(x)$.

**Addition**

The **sum** is another function $f+g$ such that $f+g(x)=f(x)+g(x)$ for $x \in \text{Domain}(f) \cap \text{Domain}(g)$

Example: $f(x)=2x-7$ $g(x)=5x+3$ $f+g(x)=2x-7+5x+3=7x-4$

**Substraction**

The **difference** is another function $f-g$ such that $f-g(x)=f(x)-g(x)$ for $x \in \text{Domain}(f) \cap \text{Domain}(g)$

Example: $f(x)=2x-7$ $g(x)=5x+3$ $f-g(x)=2x-7-(5x+3)=-3x-10$

**Multiplication**

The **product** is another function $f{\cdot}g$ such that $f{\cdot}g(x)=f(x)\cdot g(x)$ for $x \in \text{Domain}(f) \cap \text{Domain}(g)$

Example: $f(x)=2x-7$ $g(x)=5x+3$ $f\cdot g(x)=(2x-7)\cdot(5x+3)=10x^2-29x-21$

**Division**

The **quotient** is another function $\frac{f}{g}$ such that $\frac{f}{g}(x)=\frac{f(x)}{g(x)}$ for $x \in \text{Domain}(f) \cap \text{Domain}(g) \setminus g^{-1}(0)$

Example: $f(x)=2x-7$ $g(x)=5x+3$ $\frac{f}{g}(x)=\frac{2x-7}{5x+3},\quad x \neq \frac{-3}{5}$

**Composition**

The diagram shows how two functions $f$ and $g$ may be combined.

$$x \xrightarrow{f} f(x) \xrightarrow{g} g(f(x))$$

In this case, $f$ is applied first to some value $x$ giving $f(x)$. Then $g$ is applied to the value $f(x)$ to give $g(f(x))$. This is usually written as $gf(x)$ and $gf$ can be thought of as a new **composite function** defined from the functions $f$ and $g$.

**For example,** if $f(x) = 3x$, $x \in \mathbb{R}$ and $g(x) = x + 2$, $x \in \mathbb{R}$ then

$$gf(x) = g(3x) = 3x + 2.$$

The order in which the functions are applied is important. The composite function $fg$ is found by applying $g$ first and then $f$.

In this case, $fg(x) = f(x + 2) = 3x + 6$.

Generally speaking, when two functions $f$ and $g$ are defined, the composite functions $fg$ and $gf$ will not be the same.

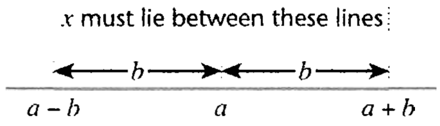
Note: When evaluating the resulting function of an operation it is possible to evaluate each function individually and then combine the two values. However, it is usually a more expedient method to combine the two functions and then do the evaluation.

## The modulus function

The notation $|x|$ is used to stand for the modulus of $x$. This is defined as

$$|x| = \begin{cases} x & \text{when } x \geqslant 0 \quad \text{(when } x \text{ is positive, } |x| \text{ is just the same as } x). \\ -x & \text{when } x < 0 \quad \text{(when } x \text{ is negative, } |x| \text{ is the same as } -x). \end{cases}$$

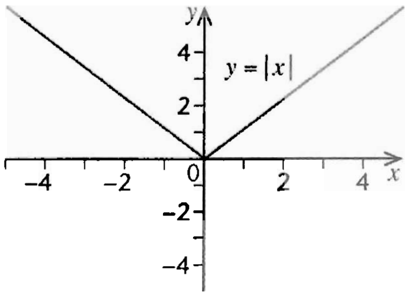
The expression $|x-a|$ can be interpreted as the distance between the numbers $x$ and $a$ on the number line. In this way, the statement $|x-a| < b$ means that the distance between $x$ and $a$ is less than $b$.




It follows that $a-b < x < a+b$.

### $y = |x|$

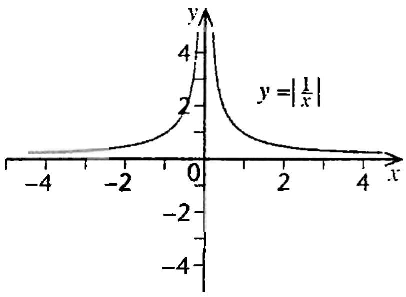
It follows that the graph of $y = |x|$ is the same as the graph of $y = x$ for positive values of $x$. But, when $x$ is negative, the corresponding part of the graph of $y = x$ must be reflected in the $x$-axis to give the graph of $y = |x|$.




### $y = |f(x)|$

The graph of $y = |f(x)|$ is the same as the graph of $y = f(x)$ for positive values of $f(x)$. But, when $f(x)$ is negative, the corresponding part of the graph of $y = f(x)$ must be reflected in the $x$-axis to give the graph of $y = |f(x)|$.

The diagram shows the graph of $y = \left|\frac{1}{x}\right|$.




## Transformations of functions

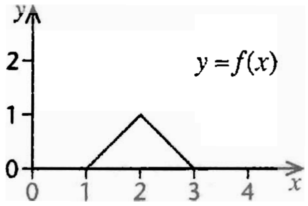
Some operations involving constant functions have a certain effect on the graph of the function.

| Known function | New function | Transformation |
|---|---|---|
| $y = f(x)$ | $y = f(x) + a$ | Translation through $a$ units parallel to $y$-axis. |
| | $y = f(x-a)$ | Translation through $a$ units parallel to $x$-axis. |
| | $y = af(x)$ | One-way stretch with scale factor $a$ parallel to the $y$-axis. |
| | $y = f(ax)$ | One-way stretch with scale factor $\frac{1}{a}$ parallel to the $x$-axis. |

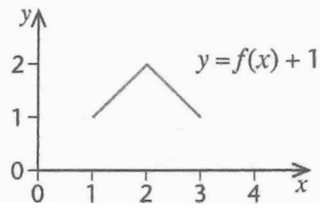
**Example** The diagram shows the graph of a function, $y=f(x)$ for $1 \leqslant x \leqslant 3$.

Use the same axes to show:

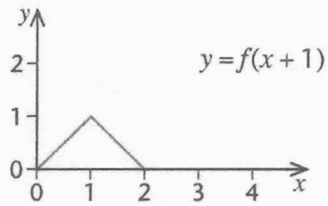
(a) $y=f(x)+1$
(b) $y=f(x+1)$
(c) $y=2f(x)$
(d) $y=f(2x)$



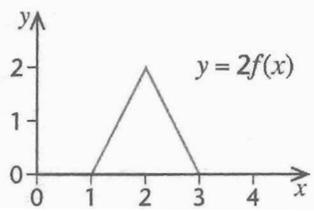

(a)



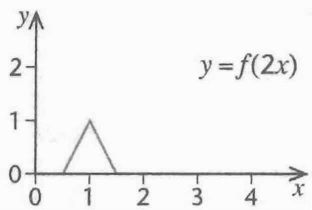

(b)




(c)




(d)




The graph of some new function can often be obtained from the graph of a known function by applying a transformation. A summary of the standard transformations is given in the table.

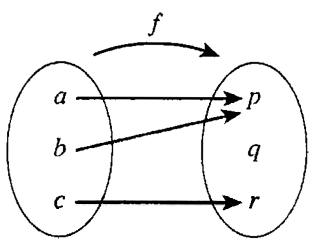
## Inverse of a function

The **inverse of a function** $f$ is a function, usually written as $f^{-1}$, that *undoes* the effect of $f$. So the inverse of a function which adds 2 to every value, for example, will be a function that subtracts 2 from every value.

This can be written as $f(x)=x+2,\ x \in \mathbb{R}$ and $f^{-1}(x)=x-2,\ x \in \mathbb{R}$.
The domain of $f^{-1}$ is given by the range of $f$.
Notice that $f^{-1}f(x)=f^{-1}(x+2)=x$ and that $ff^{-1}(x)=f(x-2)=x$.

A function can be either one–one or many–one, but only functions that are one–one can have an inverse. The reason is, that reversing a many–one function would give a mapping that is one–many, and this cannot be a function.

The diagram shows a many–one function. It does not have an inverse.

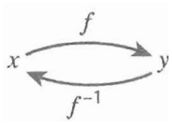



If you reverse this diagram then $p$ would have more than one image.

You can turn a many–one function into a one–one function by restricting its domain.

## Finding the inverse of a function

One way to find the inverse of a one–one function $f$ is to write $y=f(x)$ and rearrange this to make $x$ the subject so that $x=f^{-1}(y)$. The inverse function is usually defined in terms of $x$ to give $f^{-1}(x)$. The domain of $f^{-1}$ is the range of $f$




**Example** Find the inverse of the function $f(x)=\dfrac{x+2}{x-3}$, $x \neq 3$.

Define $y=\dfrac{x+2}{x-3}$

then $y(x-3)=x+2$

Multiply out the brackets.

$$xy-3y=x+2$$

$$xy-x=3y+2$$

$$x(y-1)=3y+2$$

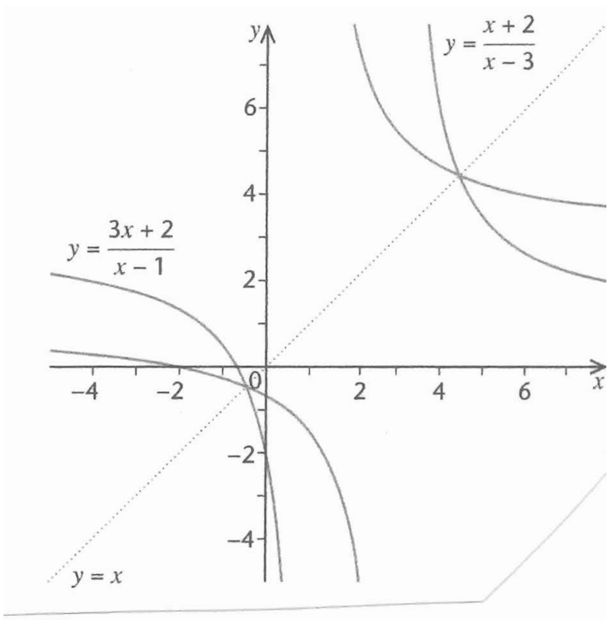
$$x=\frac{3y+2}{y-1}.$$

$x=f^{-1}(y)$ this define the inverse of f

So $f^{-1}(x)=\dfrac{3x+2}{x-1}$, $x \neq 1$.

The denominator cannot be allowed to be zero

Here are the graphs of $y=\dfrac{x+2}{x-3}$ and its inverse $y=\dfrac{3x+2}{x-1}$



## EXERCISES

20) $f(x) = x^2 + 5$ and $g(x) = 2x + 3$

(a) Write $fg(x)$ in terms of $x$. (b) Find $fg(10)$.

(c) Find the values of $x$ for which $fg(x) = gf(x)$.

21) Find the inverse of the function $f(x) = \dfrac{x+5}{x-2}$, $x \in \mathbb{R}$, $x \neq 2$ and state the domain of the inverse function.

22) (a) Sketch $y = (x-2)(x+2)$.

(b) Sketch $y = |x^2 - 4|$.

23) Solve $|x - 2| < 5$.

24) The diagram shows $y = f(x)$

On separate diagrams, sketch

(a) $y = f(x) + 1$

(b) $y = f(x+1)$

(c) $y = f(2x)$

(d) $y = 2f(x)$

(e) $y = f(\frac{1}{2}x)$