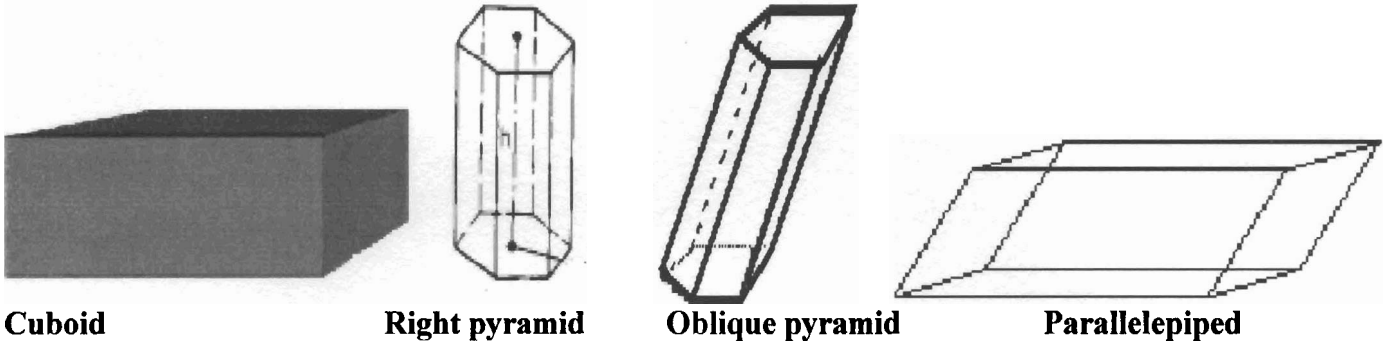Solid geometric figures

Polyhedra
(A **polyhedron** is a solid with no curved surfaces or edges. All **faces** are polygons and all **edges** are line segments. The corner points where three or more adjacent sides meet are called **vertices**).
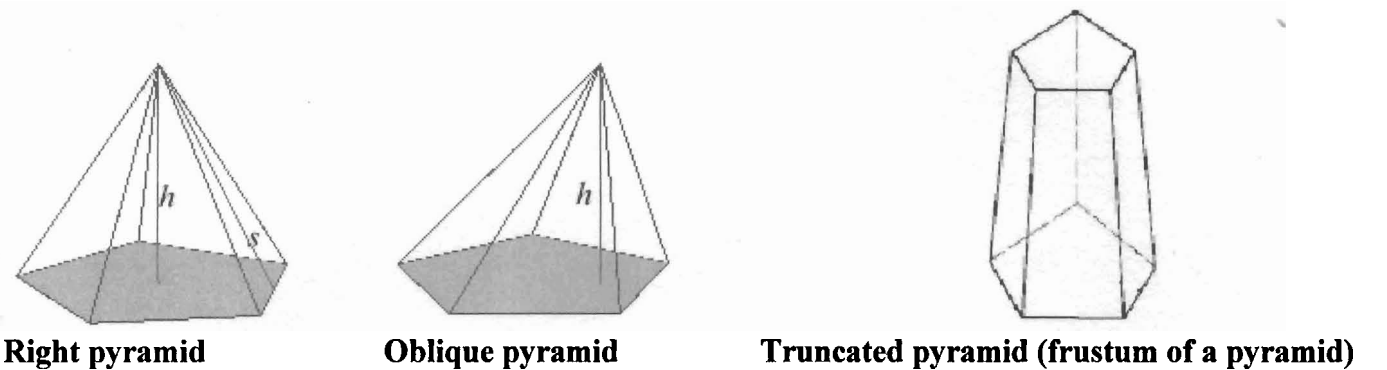
Prisms:

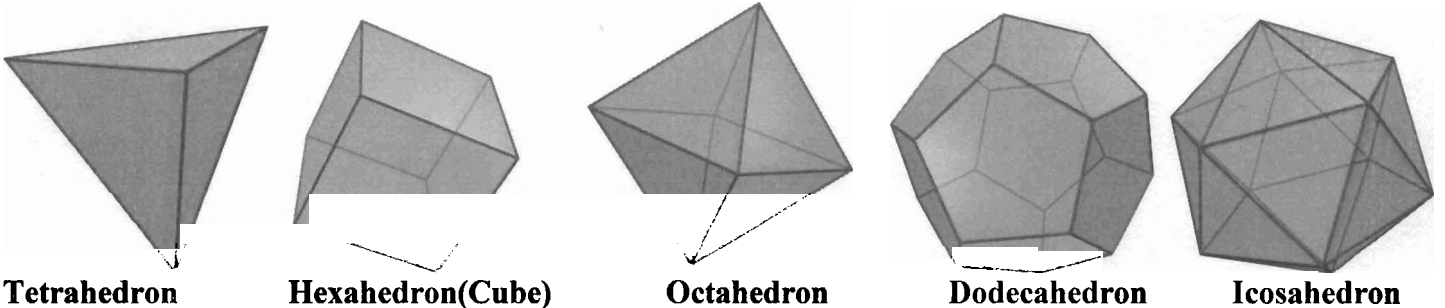**Cuboid**   **Right pyramid**   **Oblique pyramid**   **Parallelepiped**

Pyramids:

Right pyramid   **Oblique pyramid**   **Truncated pyramid (frustum of a pyramid)**

Regular polyhedra (Platonic solids)

Tetrahedron   **Hexahedron(Cube)**   **Octahedron**   **Dodecahedron**   **Icosahedron**

Every polyhedron has a **dual polyhedron** with faces and vertices interchanged. The dual of every Platonic solid is another Platonic solid, so that we can arrange the five solids into dual pairs:
- The tetrahedron is self-dual (i.e. its dual is another tetrahedron).
- The cube and the octahedron form a dual pair.
- The dodecahedron and the icosahedron form a dual pair.

One can construct the dual polyhedron by taking the vertices of the dual to be the centers of the faces of the original figure. The edges of the dual are formed by connecting the centers of adjacent faces in the original. In this way, the number of faces and vertices is interchanged, while the number of edges stays the same.

Euler's Formula (Polyhedra):   (number of faces) + (number of vertices) − (number of edges) = 2
This formula is true for all convex polyhedra as well as many types of concave polyhedra.

Solids of revolution

(The **solids of revolution** are obtained by rotating a plane figure in space about an axis coplanar to the figure).
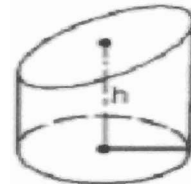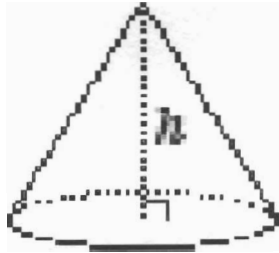
Cylinder
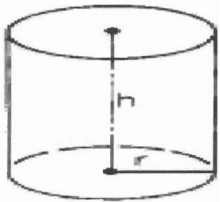
Right cylinder   **Oblique cylinder**   **Truncated cylinder**
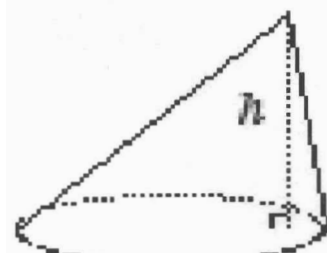
Cone
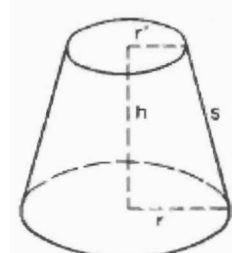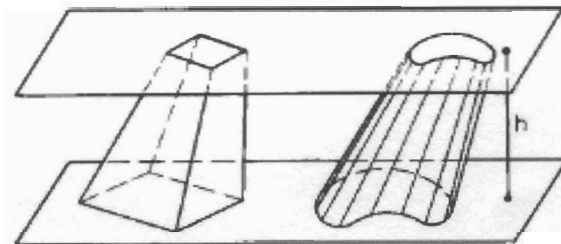
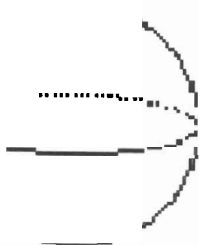**Right cone**   **Oblique cone**   **Truncated cone**   **Frustum of a cone**
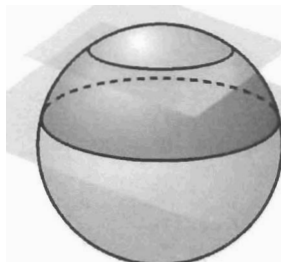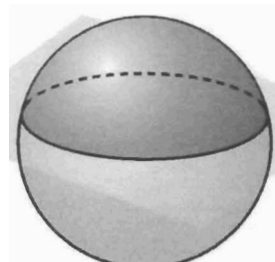
Pyramidal frustum and conical frustum.

Sphere
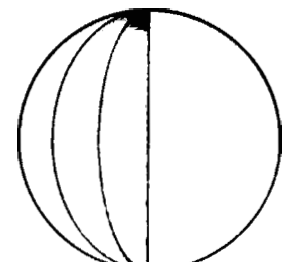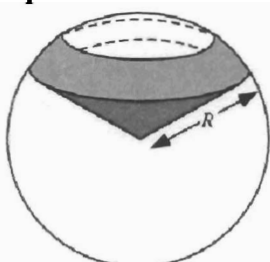
Sphere   **Spherical segment**   **Spherical cap**   **Spherical wedge**

Open spherical sector   **Spherical sector (spherical cone)**
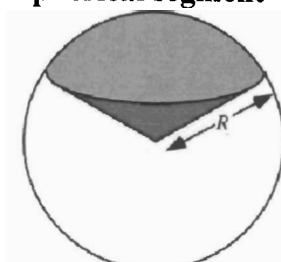
Ellipsoid

Cube: $V = a^3$
Cuboid: $V = length \cdot width \cdot height$
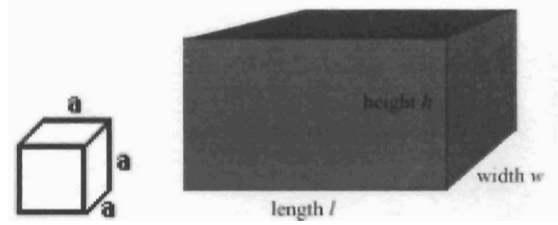
Right prism (regular and irregular): $V = b \cdot l$
Oblique prism (regular and irregular): $V = b \cdot h$
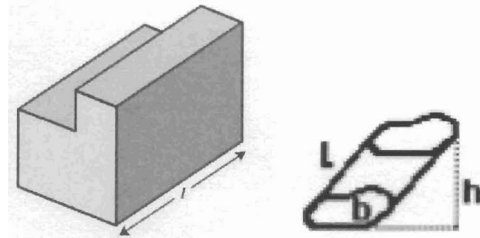
Right cylinder: $V = b \cdot l = \pi \cdot r^2 \cdot l$
Oblique cylinder: $V = b \cdot h = \pi \cdot r^2 \cdot h$
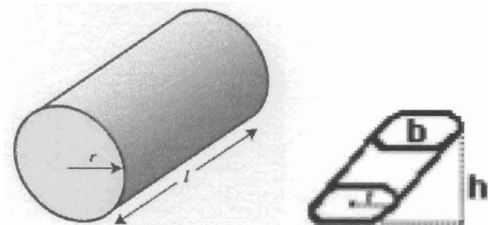
Right pyramid (regular and irregular): $V = \dfrac{b \cdot h}{3}$

Oblique pyramid (regular and irregular): $V = \dfrac{b \cdot h}{3}$
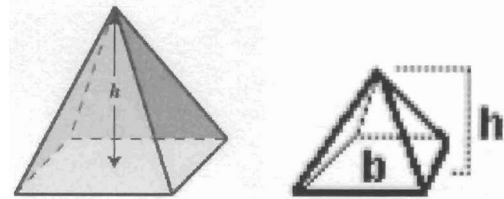
Right cone: $V = \dfrac{\pi \cdot r^2 \cdot h}{3}$

Oblique cone: $V = \dfrac{\pi \cdot r^2 \cdot h}{3}$
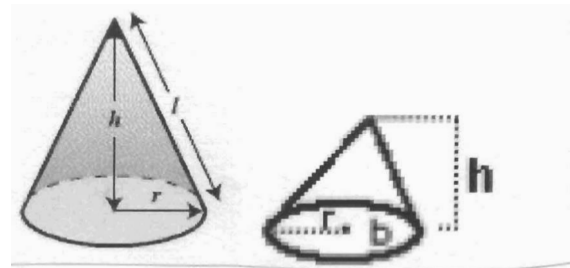
Frustum of a pyramid: $V = \dfrac{h \cdot \left(B_1 + \sqrt{B_1 \cdot B_2} + B_2\right)}{3}$

Frustum of a cone: $V = \dfrac{\pi \cdot h \cdot \left(R^2 + Rr + r^2\right)}{3}$
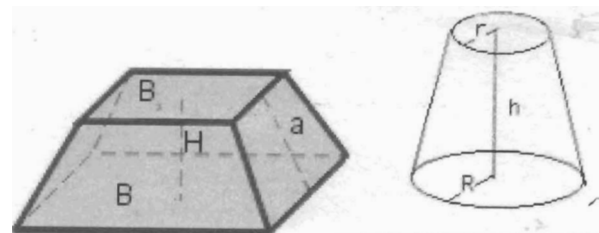
Sphere: $V = \dfrac{4 \cdot \pi \cdot r^3}{3}$

Ellipsoid: $V = \dfrac{4 \cdot \pi \cdot r_1 \cdot r_2 \cdot r_3}{3}$
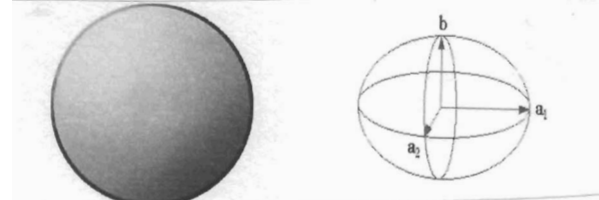
Sphere, cone and cylinder

Archimedes determined the ratio of the volume of a sphere to the volume of the circumscribed cylinder:
The volume of a cylinder is three times the volume of a cone with equal height and radius.
The volume of a sphere is two times the volume of a cone with equal height and radius.

Prove: if we consider solids with half-the volumes we can draw a semisphere, a cone with half the
                height and equal base, and a cylinder with half the height and equal base:
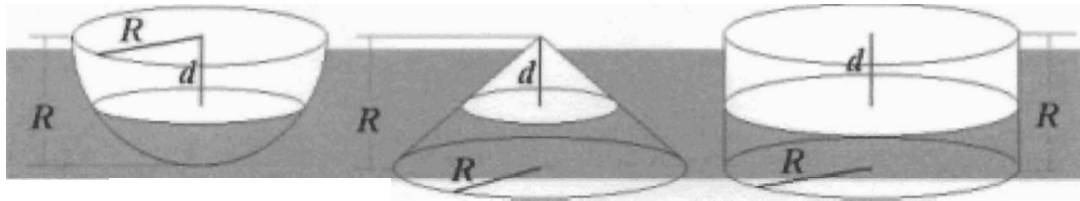
The horizontal section of half-the-cylinder is: $\boxed{\pi \cdot R^2}$ $\qquad \Rightarrow \qquad$ the volume of it is $\boxed{\pi \cdot R^3}$

The horizontal section of half-the-cone is: $\boxed{\pi \cdot d^2}$ as $\dfrac{R}{R} = \dfrac{d}{radius}$ $\Rightarrow$ the volume of it is $\boxed{\dfrac{\pi \cdot R^3}{3}}$

The horizontal section of half-the-sphere is: $\boxed{\pi \cdot \left(R^2 - d^2\right)}$ as $\boxed{radius = \sqrt{R^2 - d^2}}$ $\qquad \Rightarrow$

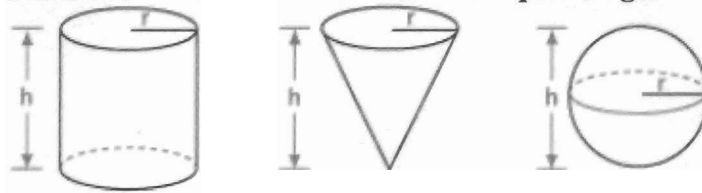$\qquad\qquad$ the volume of half-the sphere must be the difference of volumes, that is $\boxed{\dfrac{2 \cdot \pi \cdot R^3}{3}}$

$\qquad$ Conclusion: the ratio of volumes for the cylinder, cone and sphere is $\dfrac{1}{1} : \dfrac{1}{3} : \dfrac{2}{3}$ or better $3 : 1 : 2$

Areas

Planes figures

Triangle: $A = \dfrac{b \cdot h}{2} = \sqrt{p \cdot (p-a) \cdot (p-b) \cdot (p-c)}$

Parallelogram: $A = b \cdot h$

Trapezium: $A = \dfrac{(a+b) \cdot h}{2}$

Regular polygon: $A = \dfrac{n \cdot s \cdot r}{2} = \dfrac{n \cdot \sin\left(\dfrac{360°}{n}\right) \cdot R^2}{2}$

Ellypse: $A = \pi \cdot r_1 \cdot r_2$

Circle: $A = \pi \cdot r^2$

Circular sector: $A = \dfrac{\pi \cdot r^2 \cdot \theta}{2}$ $\qquad$ (θ in radians)

Annulus: $A = \pi \cdot \left(R^2 - r^2\right)$
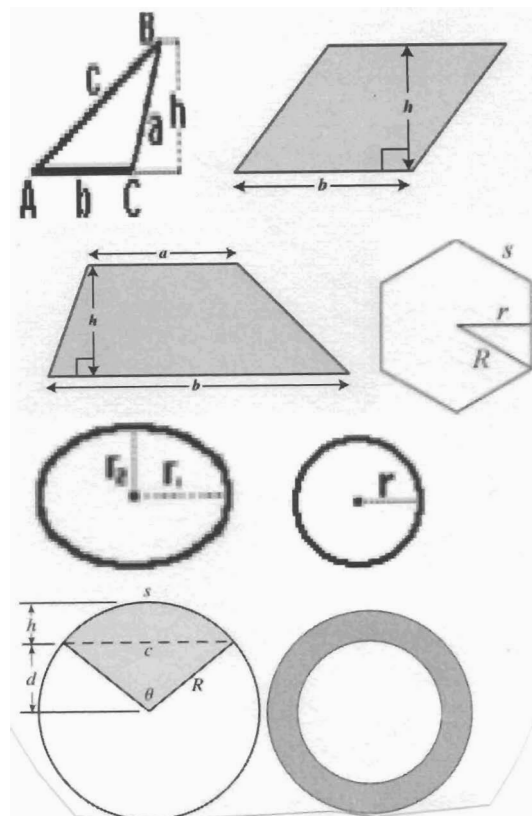
Solids

Cube: $A = 6 \cdot a^2$

Prism: $A = Lateral + 2 \cdot B = P \cdot l + 2 \cdot B$

Cylinder: $A = Lateral + 2 \cdot B = 2 \cdot \pi \cdot r \cdot l + 2 \cdot \pi \cdot r$

Pyramid: $A = Lateral + B = \dfrac{P \cdot s}{2} + B$

Frustum of a pyramid: $A = \dfrac{(P_1 + P_2) \cdot s}{2} + B_1 + B_2$

Cone: $A = \pi \cdot r \cdot s + \pi \cdot r^2 = \pi \cdot r \cdot \sqrt{r^2 + h^2} + \pi \cdot r^2$

Frustum of a cone: $A = \pi \cdot (R+r) \cdot \sqrt{h^2 + (R-r)^2} + \pi(R^2 + r^2)$

Sphere: $A = 4 \cdot \pi \cdot r^2$

Spherical segment: $A = 2 \cdot \pi \cdot R \cdot h$

Spherical wedge: $A = 2 \cdot r^2 \cdot \theta$ (θ in radians)

Spherical cap: $A = 2 \cdot \pi \cdot R \cdot h = \pi \cdot (a^2 + h^2)$

<u>Transformations of the plane</u>

In order to automate the process of animation a computer needs a formula for computing the new location of a point. In other words, the computer requires a special kind of function which takes in the coordinates of a point, and spits out the coordinates of a new, transformed point. In math, such functions are called **transformations**.

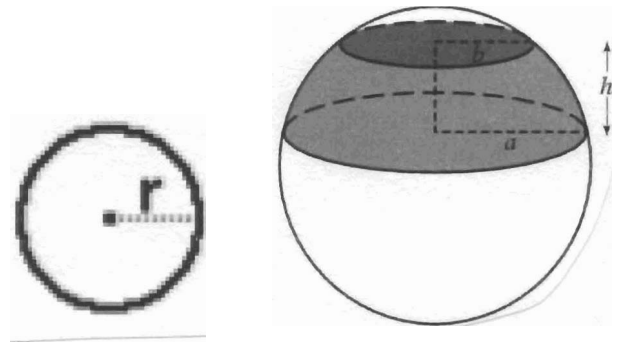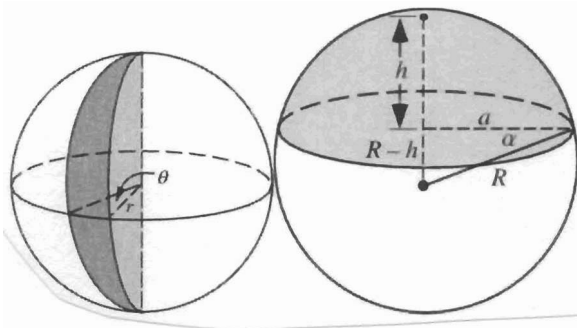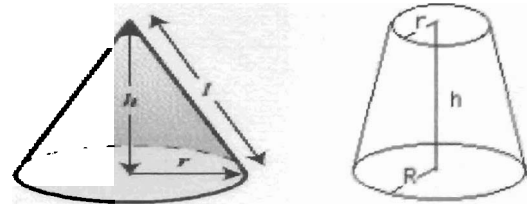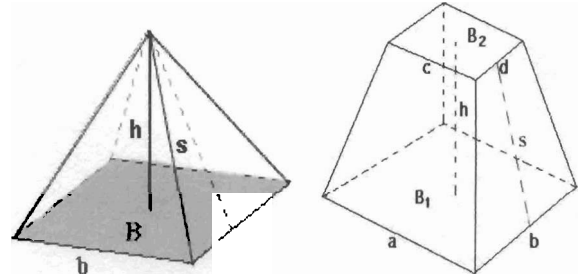In mathematics or computer graphics, picking a transformation at random is almost certain to distort objects as they move. Consequently, for computer animation, the transformations we are most interested in are the distortion-free ones, which correspond to motion in the physical world. Such transformations are called **isometries**, coming from the Greek, meaning roughly equal measures.

<u>Isometries of the plane</u>

Isometries create patterns that help us organize our world conceptually. These patterns occur in nature, and are invented by artists, craftspeople, musicians, choreographers, and mathematicians.
Plane isometry involves moving all points around the plane so that their positions relative to each other remain the same, although their absolute positions may change. **Isometries preserve distances, angles, sizes, and shapes**. They are "rigid motions".
In mathematics, the idea of isometry gives us a precise way to think about this subject. We will talk about **plane isometries**, those that take place on a flat plane, but the ideas generalize to **spatial isometries** too.

Rotation:To rotate an object means to turn it around. Every rotation has a center and an angle.

Translation:To translate an object means to move it without rotating or reflecting it. Every translation has a direction and a distance.

Reflection:To reflect an object means to produce its mirror image. Every reflection has a mirror line. A reflection of an "R" is a backwards "R". (But we can consider every reflection to be a special case of a glide reflection).

Glide Reflection:A glide reflection combines a reflection with a translation along the direction of the mirror line. Glide reflections are the only type of symmetry that involve more than one step.

The identity isometry: It can be considered a fifth kind. Alternatively, we can consider the identity a special case of a translation, and also a special case of a rotation.
We really have only three categories: rotations, translations, and glide reflections, which are mutually exclusive except for the identity.

Recognition of isometries

We can recognize which of these isometries we have according to whether it preserves orientation or not (**direct** and **opposite isometries**), and whether it has at least one fixed point or not, as shown in the following table (omitting the identity).

		Preserves orientation?	
		Yes	No
Fixed	Yes	Rotation	Reflection
point?	No	Translation	Glide reflection

Compositions of isometries

What happens when you do one isometry followed by another? Obviously the result is another isometry, since neither transformation introduces any distortion.

Two rotations of the same centre produce a new rotation with the same centre and with angle equal to the adition of the angles.
Two translations produce a new translation in which the vector is equal to the adition of the vectors.

If we do **one reflection followed by another,** notice the result will have to be orientation-preserving; that suggests a product of reflections should be either a translation or a rotation. (depending on whether or not the mirror lines of the two relfections cross; if the mirror lines cross the intersection point will be a fixed point of the product).
We also can emulate the effect of translation by doing **two reflections.**
We can emulate the effect of a glide reflection by doing **three reflections.**
As a result, we can really just focus on the behavior of reflections, since **the other three types of isometry can be built up from reflections.**

Any isometry of the plane is a translation, rotation, reflection or glide reflection; **no matter how you compose them, you will always end up with another isometry on this list.**

N-fold rotational symmetry

Rotational symmetry of order n, also called **n-fold rotational symmetry,** or **discrete rotational symmetry of n**[th] **order,** with respect to a particular point (in 2D) means that rotation by an angle of 360°/n does not change the object.
The notation for n-fold symmetry is C_n or simply "n".
Examples: n = 2, 180°: rectangle, rhombus, letters Z, N, S, yin and yang symbol
                    n = 3, 120°: triskelion
                    n = 4, 90°: swastika
                    n = 6, 60°: raelian symbol, new version

More examples.

Nederlandse Spoorwegen (Dutch Railways)   N=2

Traffic sign roundabout   N=3

The flag of Nazi Germany and the NSDAP, with swastika.   N=4

White circle in Syrian Social Nationalist Party flag   N=4

The swastika in decorative Hindu form   N=4

The Raëlian symbol, before and after 1991   N=2, N=6

EXERCISES

1) Classify all the capital letters in English (in their simplest forms) according to their symmetries. For example, "A" has a reflection in a vertical line, and "R" has no symmetry.

2) Make a pattern by starting with an asymmetric shape and repeating a single translation over and over. That is, decide on a direction and distance for your translation (for example, 5 cm to the right). Translate your shape 5 cm to the right, then translate the new one 5 cm to the right, etc. Also translate the original letter 5 cm. to the left, etc. Did you get any other types of isometries (reflections, glide reflections, or rotations) in the process?

3) Make a symmetric pattern by starting with an asymmetric shape and repeating a single glide reflection over and over (also glide it backwards). That is, pick a reflection line and a translation in a direction parallel to the reflection line. Keep applying the same glide reflection to the new shapes that you generate until you run out of paper. Did you get any other types of isometries (reflections, translations, or rotations) in the process?

4) On a piece of paper, draw a letter R (call it R1) and two parallel lines (L1 and L2) about an inch apart. Reflect the R across L1 and call it R2. Reflect R2 across L2 and call it R3.
   a)   How is R3 related to R1? (by which type of isometry?)
   b)   Continue your pattern by reflecting the new Rs across L1 and L2. Would this be an infinite pattern if you had an infinitely large piece of paper?
   c)   What isometries does your pattern have besides reflections across L1 and L2?

5) Repeat the previous activity using lines that are not parallel but intersect at a 45-degree angle.
   a)   Now how is R3 related to R1?
   b)   Continue your pattern by reflecting the new Rs across L1 and L2. Would this be an infinite pattern if you had an infinitely large piece of paper?
   c)   What isometries does your pattern have besides reflections across L1 and L2?

6) Do you remember making strings of paper dolls or snowflakes by cutting a strip of folded paper? Adapt one of these activities so that it explicitly talks about isometries.

7) Consider patterns generated by reflections across two intersecting lines, but use other angles. For example, what if the lines formed an angle of 111 degrees, or 60 degrees? If you can, find a general pattern that will predict results for any angle.

Symmetry in the arts and crafts

A **frieze group** is a mathematical concept to classify designs on two-dimensional surfaces which are repetitive in one direction, based on the symmetries in the pattern. Such patterns occur frequently in architecture and decorative art. The mathematical study of such patterns reveals that exactly 7 different types of patterns can occur.
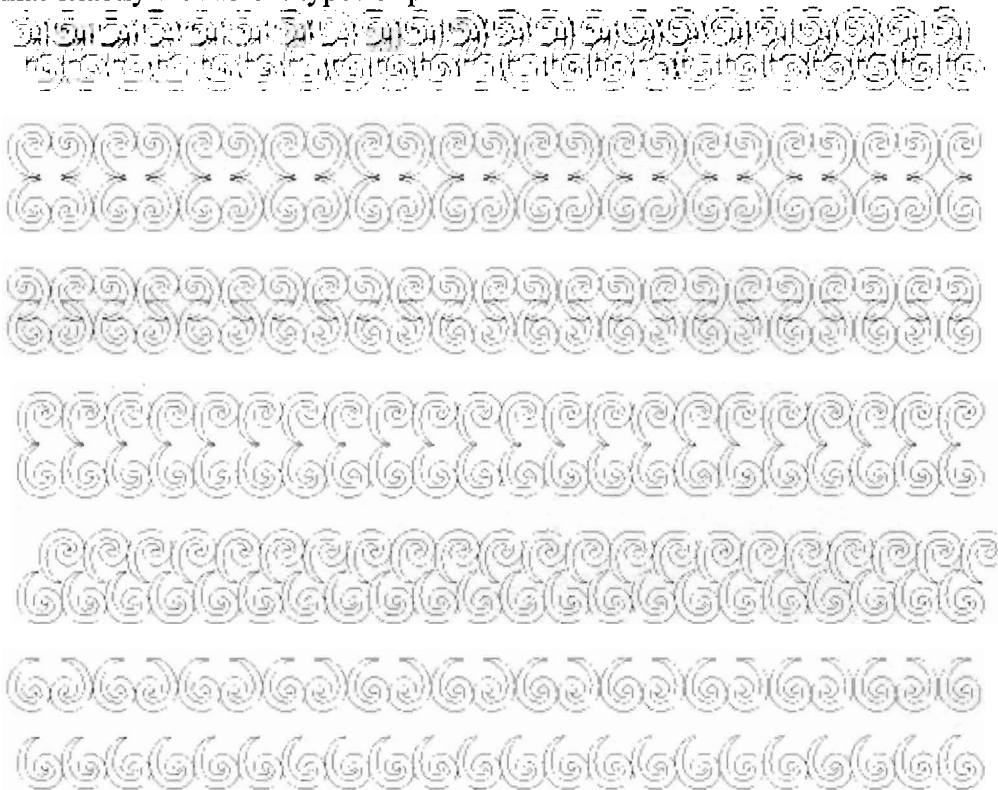
A **wallpaper group** (or plane symmetry group or plane crystallographic group) is a mathematical classification of a two-dimensional repetitive pattern, based on the symmetries in the pattern. Such patterns occur frequently in architecture and decorative art. There are 17 possible distinct groups.

Most commonly, and especially in Gothic architecture, a **rose window** is a circular stained glass window, with mullions and traceries that generally radiate from the centre. In churches and cathedrals its use sometimes alludes to the Virgin Mary, who is called the Mystical Rose; and occasionally to elaborate forms based on the rose.

You can find the use of symmetry across a wide variety of arts and crafts.

Architecture
Prominent examples include the Leaning Tower of Pisa, Monticello, the Astrodome, the Sydney Opera House, Gothic church windows, and the Pantheon. Symmetry is used in the design of the overall floor plan of buildings as well as the design of individual building elements such as doors, windows, floors, frieze work, and ornamentation; many facades adhere to bilateral symmetry.
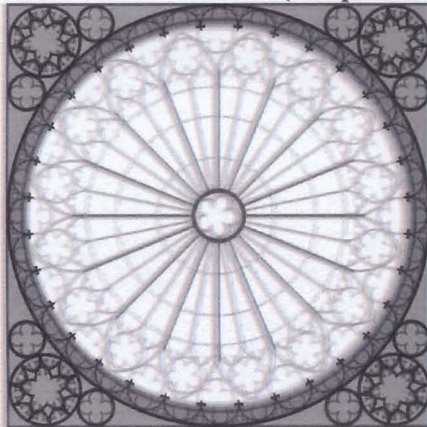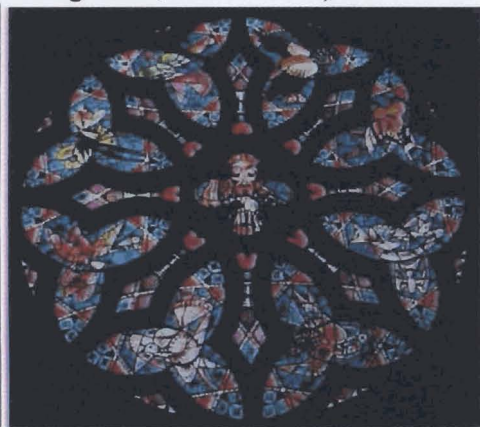
(Anthemion frieze)

(Chapelle de Languidou, Pont l'Abbé)
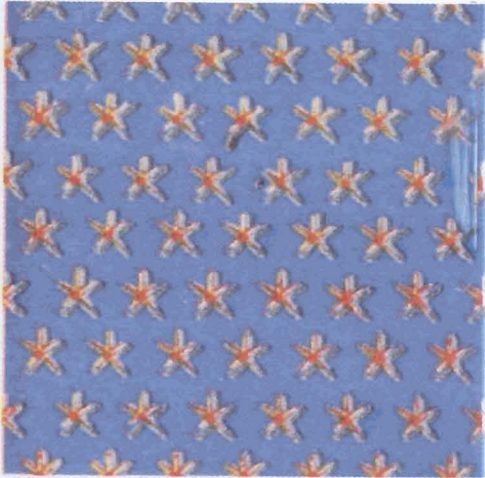
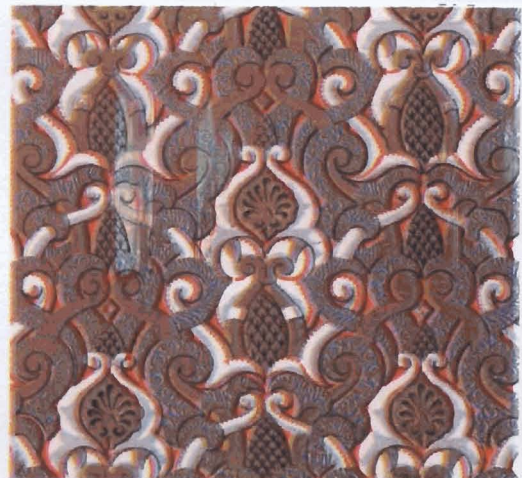(Leaning Tower of Pisa)   (Strasburg muenster)   (Michaelhouse School)

(ceilings of Egyptian tombs)   (tomb in Tebas, Egypt)

(Dado from Biban el Moluk, Egypt)

(Spandrils of arches, the Alhambra)

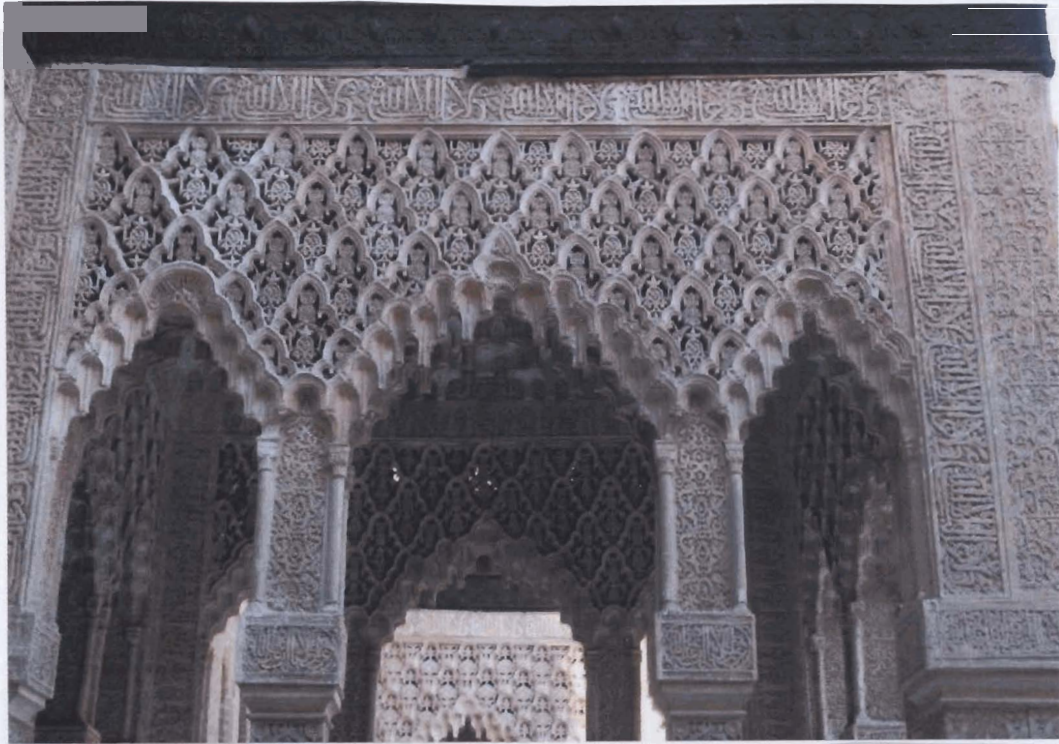(Saint-Christophe church, Couddes, Loir-et-Cher)

(Courtyard of the Lions, the Alhambra)

(Patio de los Arrayanes, the Alhambra)

(ceiling decoration, the Alhambra)

(Mocárabe arches, the Alhambra)
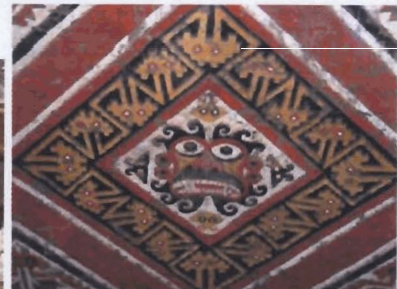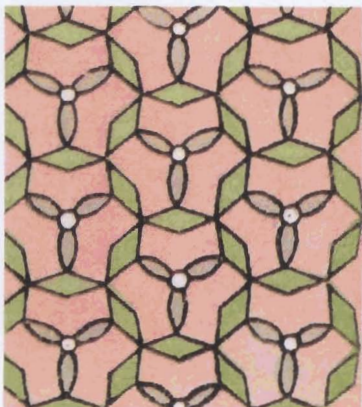

(Indian metalworks)
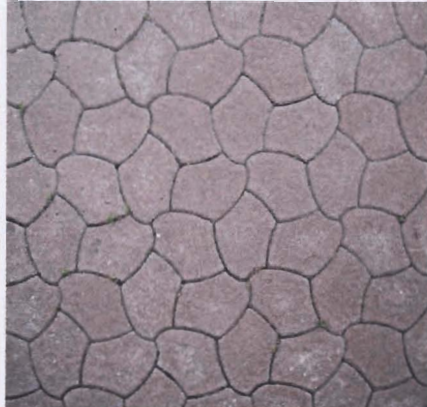

(Pazar Han, Anatolia)   (Kabah. Yucatan. Mexico)   (Huaca de la Luna, Trujillo)

(persian glazed tile)   (Street pavement, Zakopane, Poland) (Bathroom linoleum, U.S.)

(Frieze above door in the Richmond Room, Metropolitan Museum of Art, New York)

(modern metal frieze, Australia)

(Virchaux Drapery-Swan Frieze, Louisville)

(Art Deco attic frieze)

Pottery

The ancient Chinese used symmetrical patterns in their bronze castings since the 17th century
B.C. Bronze vessels exhibited both a bilateral main motif and a repetitive translated border
design. Persian pottery dating from 6000 B.C. used symmetric zigzags, squares, and cross-
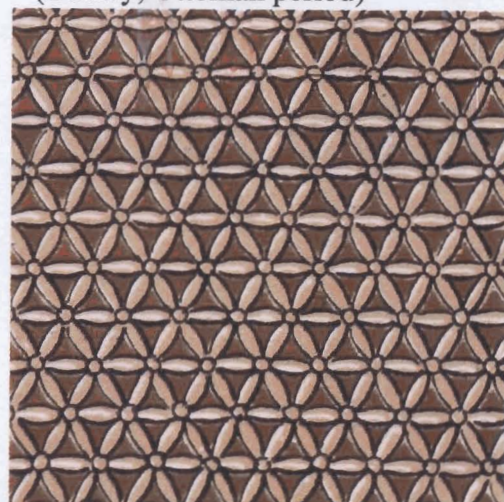hatchings.

(Painted porcelain, China)

(Turkey, Ottoman period)

(Bronze vessels in Nimroud, Assyria)

Music

Symmetry has been used as a formal constraint by many composers, such as the arch form
(ABCBA) used by Steve Reich, Béla Bartók, and James Tenney (or swell). In classical music,
Bach used the symmetry concepts of permutation and invariance; see (external link "Fugue
No. 21," pdf or Shockwave).

Symmetry is also an important consideration in the formation of scales and chords, traditional
or tonal music being made up of non-symmetrical groups of pitches, such as the diatonic scale
or the major chord. Symmetrical scales or chords, such as the whole tone scale, augmented
chord, or diminished seventh chord (diminished-diminished seventh), are said to lack
direction or a sense of forward motion, are ambiguous as to the key or tonal center, and have a
less specific diatonic functionality. However, composers such as Alban Berg, Béla Bartók,
and George Perle have used axes of symmetry and/or interval cycles in an analogous way to
keys or non-tonal tonal centers.

Quilts, carpets, rugs and clothes
As quilts are made from square blocks (usually 9, 16, or 25 pieces to a block) with each
smaller piece usually consisting of fabric triangles, the craft lends itself readily to the
application of symmetry.

Quilt Block Design

(Hood, Northern Cree, Late 19th century)

A long tradition of the use of symmetry in rug patterns spans a variety of cultures. American
Navajo Indians used bold diagonals and rectangular motifs. Many Oriental rugs have intricate
reflected centers and borders that translate a pattern. Not surprisingly most rugs use
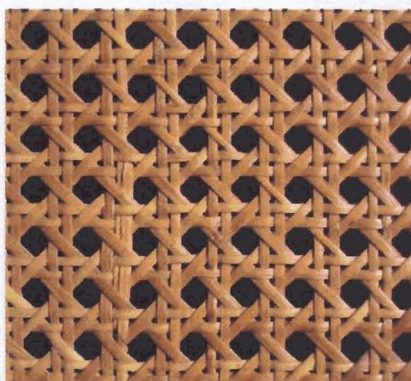quadrilateral symmetry (a motif reflected across both the horizontal and vertical axes).
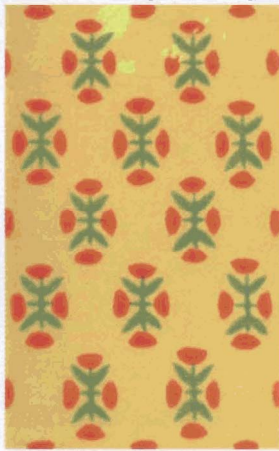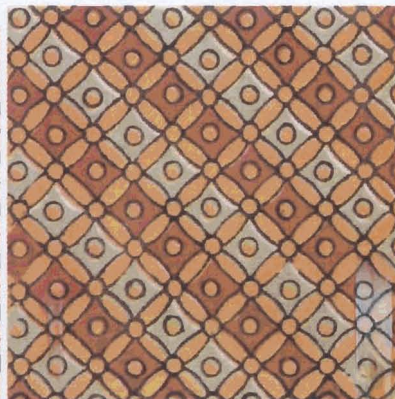

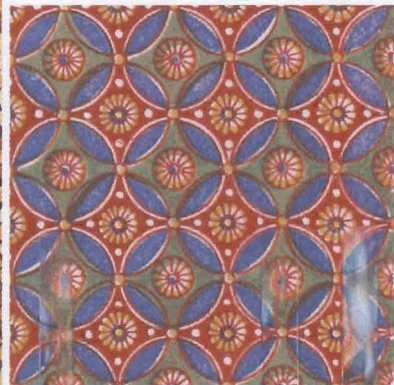(persian tapestry)

(medieval wall diapering)

(Renaissance earthernware)

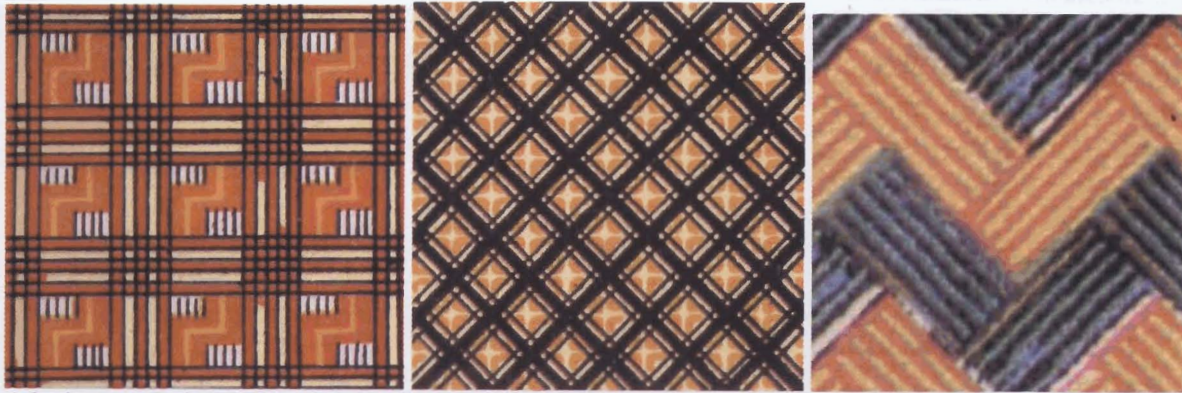(Viennese cane)

(Mummy case, the Louvre)

(Egyptian mummy case)

(cloth, Sandwich Islands, Hawaii)   (Cloth, Otaheite, Tahiti)   (Egyptian mat)

Celtic knotwork

The concept of symmetry is applied to the design of objects of all shapes and sizes -- you can find it in the design of beadwork, furniture, sand paintings, knotwork, masks, and musical instruments (to name just a handful of examples).

Aesthetics

Symmetry does not by itself confer beauty to an object — many symmetrical designs are boring or overly challenging, and on the other hand preference for, or dislike of, exact symmetry is apparently dependent on cultural background. Along with texture, color, proportion, and other factors, symmetry does however play an important role in determining the aesthetic appeal of an object.

(persian ornaments)   (Ornamental painting, Nineveh, Assyria)

(Viking broochs)

(Expectation, by Gustav Klimt)