Analytic geometry

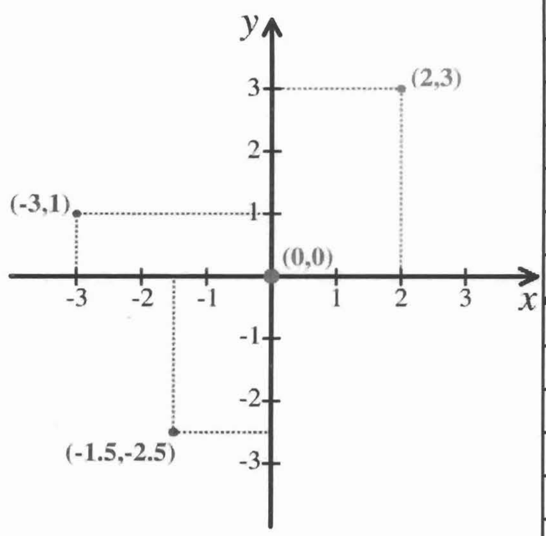
Analytic Geometry and **Coordinate Geometry** study geometric figures using the coordinate plane or coordinates in space.
Using the Cartesian coordinate system geometric shapes (such as curves) can be described by algebraic equations, namely equations satisfied by the coordinates of the points lying on the shape. For example, the circle of radius 2 may be described by the equation $x^2 + y^2 = 4$.

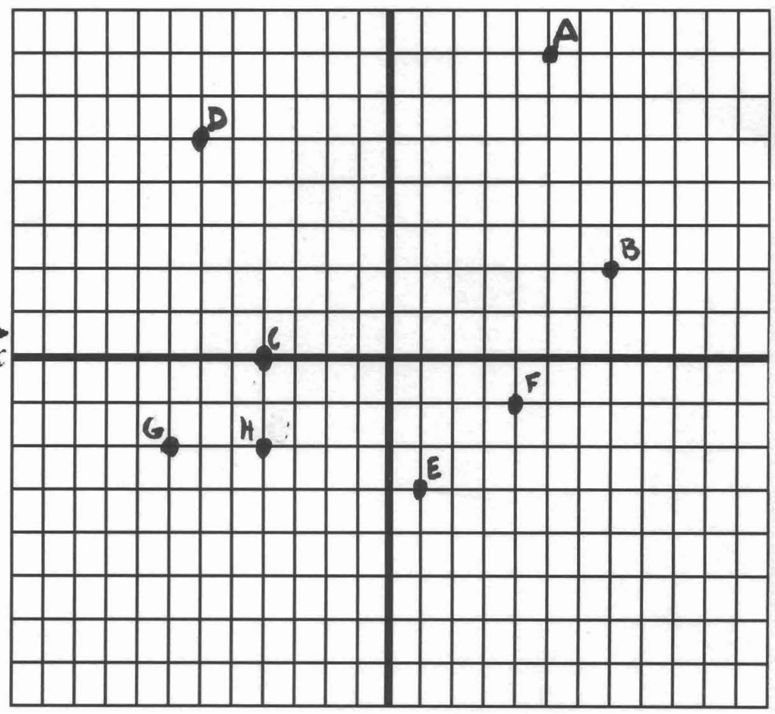
Points in the cartesian coordinate system

In mathematics, the **Cartesian coordinate system** is used to uniquely determine each **point in the plane** P(x,y) through an ordered pair of numbers, usually called the **x-coordinate** and the **y-coordinate** of the point.
To define the coordinates, two perpendicular directed lines (**the x-axis or abscissa** and the **y-axis or ordinate**), are specified, as well as **the unit length**, which is marked off on the two axes. The point at which the axes cross is called **the origin**. The standard way of orienting the axes: the positive x-axis pointing right and the positive y-axis pointing up (and the x-axis being the "first" and the y-axis the "second" axis).

Examples:

A(5,7) B(7,2) C(–4,0) D(–6,5) E(1,–3) F(4,–1) G(–7,–2)
H(–4,–2)





Note: Cartesian coordinate systems are also used in space (where three coordinates are used) and in higher dimensions.

Directed line segments

Fixed vectors are usually shown as arrows, called **directed line segments**.
If the direction is from point $A(a_x,a_y)$ to $B(b_x,b_y)$, the directed line segment is denoted by $\overline{AB}$.

The point A is called **the initial point, tail, base, start, or origin.**
The point B is called **the terminal point, head, tip, endpoint, or destination.**

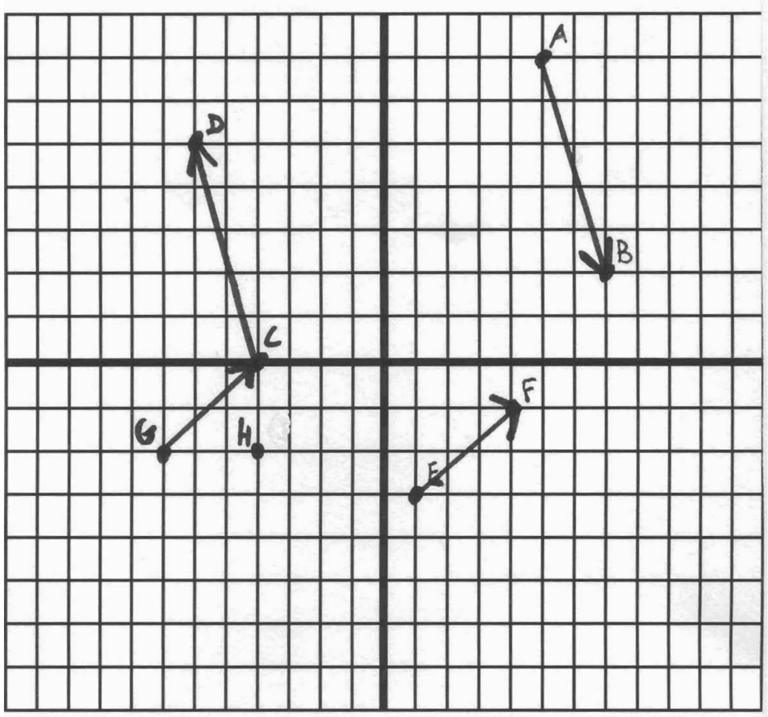
The length of the arrow represents the vector's **magnitude**:

$$\textbf{norm or magnitude}\ \left|\overrightarrow{AB}\right| = \sqrt{(b_x - a_x)^2 + (b_y - a_y)^2} \quad \text{(Pythagoras)}$$

The direction in which the arrow points represents the vector's **direction**.:

$$\textbf{slope or gradient}\ m_{AB} = \frac{b_y - a_y}{b_x - a_x} \qquad \left(\frac{rise}{run}\right)$$

Examples: $\overrightarrow{AB}$ $\overrightarrow{CD}$ $\overrightarrow{GC}$ $\overrightarrow{EF}$



$$\left|\overrightarrow{AB}\right| = \sqrt{(7-5)^2 + (2-7)^2} = \sqrt{29}$$

$$m_{AB} = \frac{2-7}{7-5} = \frac{-5}{2}$$

$$\left|\overrightarrow{CD}\right| = \sqrt{(-6+4)^2 + (5-0)^2} = \sqrt{29}$$

$$m_{CD} = \frac{5-0}{-6+4} = \frac{-5}{2}$$

$$\left|\overrightarrow{GC}\right| = \sqrt{(-4+7)^2 + (0+2)^2} = \sqrt{13}$$

$$m_{GC} = \frac{0+2}{-4+7} = \frac{2}{3}$$

$$\left|\overrightarrow{EF}\right| = \sqrt{(4-1)^2 + (-1+3)^2} = \sqrt{13}$$

$$m_{EF} = \frac{-1+3}{4-1} = \frac{2}{3}$$

$\overrightarrow{AB}$ and $\overrightarrow{CD}$ are parallel with same direction and opposite sense.

$\overrightarrow{GC}$ and $\overrightarrow{EF}$ are parallel and have equal direction and sense. They are said to be equivalent.

Let A(a_x,a_y) and B(b_x,b_y) be points in the plane; we call the **coordinates of vector** $\overrightarrow{AB}$ to

$$B - A = (b_x - a_x, b_y - a_y)$$

Two directed line segments (fixed vectors) are **equivalent** if they have the same magnitude, direction and sense. That happens when they have the same coordinates.

Let A(a_x,a_y) , B(b_x,b_y), C(c_x,c_y) and D(d_x,d_y) be points in the plane and consider $\overrightarrow{AB}$ and $\overrightarrow{CD}$

$$\overrightarrow{AB} \cong \overrightarrow{CD} \Leftrightarrow \begin{cases} b_x - a_x = d_x - c_x \\ \text{and} \\ b_y - a_y = d_y - c_y \end{cases} \Leftrightarrow \overrightarrow{AB} \text{ and } \overrightarrow{CD} \text{ have the same coordinates}$$

Example: $\overrightarrow{AB}(2,-5)$ $\overrightarrow{CD}(-2,5)$ $\overrightarrow{GC}(3,2)$ $\overrightarrow{EF}(3,2)$

Free vectors

Each class of equivalent fixed vectors determines a **free vector**. The **coordinates of the free vector** are those of all the fixed vectors of the class. Each fixed vector of the class is said to be an instance of the free vector. $\overrightarrow{PQ}(v_x, v_y)$ is an instance of free vector $\vec{v}(v_x, v_y)$

Examples: $\vec{u}(2,-5), \vec{v}(-2,5), \vec{w}(3,2)$

$\overrightarrow{AB}$ is an instance of $\vec{u}$

$\overrightarrow{CD}$ is an instance of $\vec{v}$

$\overrightarrow{GC}$ and $\overrightarrow{EF}$ are instances of $\vec{w}$

A **free vector** consists of a magnitude and a direction (direction and sense).

Vectors are usually denoted in boldface, as **v**; other conventions include $\vec{v}$ or v̲, especially in handwriting; alternately, some use a tilde (~) or a wavy underline drawn beneath the symbol, which is a convention for indicating boldface type.

Vectors are usually shown in graphs or other diagrams as arrows.

The length of the arrow represents the vector's magnitude, while the direction in which the arrow points represents the vector's direction.

The length of the arrow represents the vector's **magnitude**:

norm or **magnitude** of $\vec{v}(v_x, v_y)$ $|\vec{v}| = \sqrt{v_x^2 + v_y^2}$ (Pythagoras)

The direction in which the arrow points represents the vector's **direction**.:

slope or **gradient** of $\vec{v}(v_x, v_y)$ $m_{\vec{v}} = \frac{v_y}{v_x}$ $\left(\frac{rise}{run}\right)$

Examples:

$|\vec{u}| = \sqrt{2^2 + (-5)^2} = \sqrt{29}$ $m_{\vec{u}} = \frac{-5}{2}$

$|\vec{v}| = \sqrt{(-2)^2 + 5^2} = \sqrt{29}$ $m_{\vec{v}} = \frac{-5}{2}$

$|\vec{w}| = \sqrt{3^2 + 2^2} = \sqrt{13}$ $m_{\vec{w}} = \frac{2}{3}$

Note: Any vector **v** can be written as **v** = v_x**i** + v_y**j** with real numbers v_x and v_y which are uniquely determined by **v** (**i** and **j** are the unit vectors parallel to the *x*-and *y*-axes respectively).

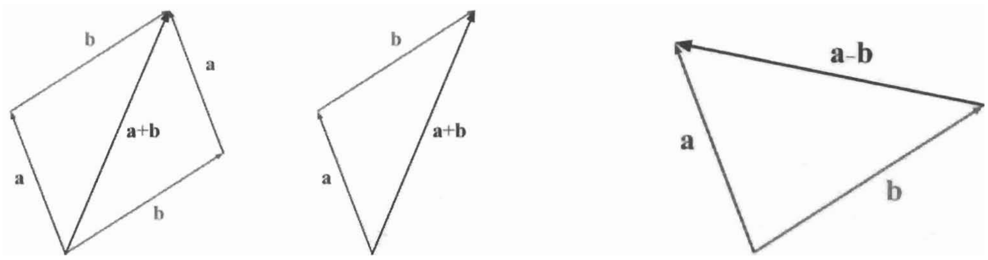
Note: Whenever we say 'vector', without specifying free or fixed, we mean 'free vector'

Note: If we add separately x-coordinates and y-coordinates we can see that $P + PQ = Q$, so when placed at *P*, $\overrightarrow{PQ}$ decribes how to get from *P* to *Q*.

Vector addition and subtraction

Given the vectors $\vec{a}(a_x, a_y)$ and $\vec{b}(b_x, b_y)$ the **sum** of $\vec{a}$ and $\vec{b}$ is $\vec{a}+\vec{b}=(a_x+b_x, a_y+b_y)$

Given the vectors $\vec{a}(a_x, a_y)$ and $\vec{b}(b_x, b_y)$ the **difference** of $\vec{a}$ and $\vec{b}$ is $\vec{a}-\vec{b}=(a_x-b_x, a_y-b_y)$
The addition may be represented graphically by placing the start of the arrow **b** at the tip of the arrow **a**, and then drawing an arrow from the start of **a** to the tip of **b**. The new arrow drawn represents the vector **a** + **b**, as illustrated below. This addition method is sometimes called the **parallelogram rule** because **a** and **b** form the sides of a parallelogram and **a** + **b** is one of the diagonals.
Subtraction of two vectors can be geometrically defined as follows: to subtract **b** from **a**, place the ends of **a** and **b** at the same point, and then draw an arrow from the tip of **b** to the tip of **a**. That arrow represents the vector **a** − **b**, as illustrated below.



Scalar multiplication

Given the vector $\vec{a}(a_x, a_y)$ and the real number $\lambda \in \Re$ the **product** of $\vec{a}$ by λ is $\lambda\vec{a}=(\lambda a_x, \lambda a_y)$
The length of *r***a** is |λ||**a**|. If the scalar is negative, it also changes the direction of the vector by 180°.
Two examples (λ = -1 and λ = 2) are given in the picture.
Here it is important to check that the scalar multiplication is compatible with vector addition in the following sense: λ(**a** + **b**) = λ**a** + λ**b** for all vectors **a** and **b** and all scalars λ.
One can also show that **a** - **b** = **a** + (-1)**b**.

Note: Multiplication of a non-zero free vector by a real number has the following effects:

$k < -1$	stretch and reverse direction
$k = -1$	reverse direction
$-1 < k < 0$	shrink and reverse direction
$k = 0$	becomes zero vector
$0 < k < 1$	shrink
$k = 1$	do nothing
$k > 1$	stretch

Note: In mathematics, numbers are often called **scalars** to distinguish them from vectors.

Dot product

Given the vectors $\vec{a}(a_x, a_y)$ and $\vec{b}(b_x, b_y)$ the **dot product** (sometimes called **inner product** or **scalar product**) of $\vec{a}$ and $\vec{b}$ is $\vec{a}\cdot\vec{b}=|\vec{a}|\cdot|\vec{b}|\cdot\cos\theta$. It can be also defined as $\vec{a}\cdot\vec{b}=a_x\cdot b_x+a_y\cdot b_y$ where $|\vec{a}|$ and $|\vec{b}|$ denote the norm (or length) of $\vec{a}$ and $\vec{b}$ and θ is the measure of the angle between $\vec{a}$ and $\vec{b}$.
Geometrically, this means that $\vec{a}$ and $\vec{b}$ are drawn with a common start point and then the length of $\vec{a}$ is multiplied with the length of that component of $\vec{b}$ that points in the same direction as $\vec{a}$.

Unit vector

A **unit vector** is any vector with a length of one.
If you have a vector of arbitrary length, you can use it to create a unit vector. This is known as **normalizing** a vector.

To normalize a vector $\vec{a}(a_x, a_y)$, scale the vector by the reciprocal of its length $|\vec{a}|$. That is: $\vec{u}_a = \frac{\vec{a}}{|\vec{a}|}$

Relative positions of vectors and points

Parallel vectors

$\vec{u}(u_x, u_y)$ and $\vec{v}(v_x, v_y)$ are said to be **parallel** if $\vec{v} = \lambda \cdot \vec{u}$ for some $\lambda \in \Re$. It is written $\vec{u} \parallel \vec{v}$ in concise form.

$$\vec{u} \parallel \vec{v} \Leftrightarrow \left\{ \begin{matrix} v_x = \lambda \cdot u_x \\ v_y = \lambda \cdot u_y \end{matrix} \right\} for \lambda \in \Re \Leftrightarrow \frac{v_x}{u_x} = \frac{v_y}{u_y} \Leftrightarrow \frac{u_y}{u_x} = \frac{v_y}{v_x} \Leftrightarrow m_{\vec{u}} = m_{\vec{v}}$$

(parallel) (proportional coordinates) (equal slopes)

Examples: $\vec{u}(8,-12)$ and $\vec{v}(10,-15)$ are parallel because both of them have slope $m = -\frac{3}{2}$

$\vec{u}(1,4)$ and $\vec{v}(3,8)$ are not parallel because they have slopes $m_{\vec{u}} = 4$ and $m_{\vec{v}} = \frac{8}{3}$

Note: The angle between two parallel vectors is 0° or 180°.

Perpendicular vectors

$\vec{u}(u_x, u_y)$ and $\vec{v}(v_x, v_y)$ are said to be **perpendicular** if $\vec{v} \cdot \vec{u} = 0$. It is written $\vec{u} \perp \vec{v}$ in concise form.

$$\vec{u} \perp \vec{v} \Leftrightarrow u_x \cdot v_x + u_y \cdot v_y = 0 \Leftrightarrow \frac{v_y}{v_x} = -\frac{u_x}{u_y} \Leftrightarrow m_{\vec{v}} = \frac{-1}{m_{\vec{u}}} \Leftrightarrow m_{\vec{u}} \cdot m_{\vec{v}} = -1$$

(perpendicular) (slope$_1$ is minus the reciprocal of slope$_2$)(product of slopes is –1)

Examples: $\vec{u}(8,-12)$ and $\vec{v}(3,2)$ are perpendicular because their slopes are $m_{\vec{u}} = \frac{-3}{2}$ and $m_{\vec{v}} = \frac{2}{3}$

$\vec{u}(4,-12)$ and $\vec{v}(3,6)$ are not perpendicular because their slopes are $m_{\vec{u}} = -3$ and $m_{\vec{v}} = 2$

Note: The angle between two perpendicular vectors is 90° or 270°.

Aligned points

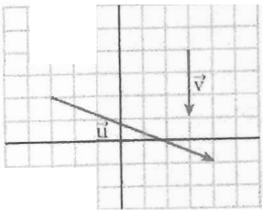
We can check if points A, B and C in the plane are in the same line calculating the coordinates of vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$; if those vectors are parallel, then the points are **aligned**.

Examples: A(–1,0) B(0,4) C(5,5) are not aligned because $\overrightarrow{AB}(1,4)$ and $\overrightarrow{BC}(5,1)$ are not parallel

A(–1,0) B(0,4) C(0.5,6) are aligned because $\overrightarrow{AB}(1,4)$ and $\overrightarrow{BC}(0.5,2)$ are parallel

EXERCISES

1) Given A(–2,0) B(0,4) C(5,2) D(3,–4) calculate:
a) The magnitude of the following fixed vectors: $\overrightarrow{AB}$ $\overrightarrow{BC}$ $\overrightarrow{CD}$ $\overrightarrow{BA}$ $\overrightarrow{DB}$
b) The gradient of the same vectors:
c) The coordinates of the vectors:

2) 

a) Say the coordinates of vectors $\vec{u}$ and $\vec{v}$
b) Draw vector $\vec{u}+\vec{v}$ and say its coordinates

3) Given vectors $\vec{u}(4,-2)$ and $\vec{v}(-2,-1)$:

a) Draw the following: $\vec{u}+\vec{v}$ $\vec{u}-\vec{v}$ $\frac{1}{2}\vec{u}$ $-3\vec{v}$

b) Write the coordinates of $\vec{w}=2\vec{u}+3\vec{v}$

4) Let $\vec{u}(3,2)$, $\vec{v}(x,5)$ and $\vec{w}(4,y)$ be three free vectors.
Calculate "x" and "y" to make true the equality $2\vec{u}-\vec{v}=\vec{w}$

5)
a) Write the magnitude and gradient of $\overrightarrow{PQ}$ where P(2 , -16) and Q(3, -5). Calculate its coordinates.
b) Write the magnitude and gradient of $\overrightarrow{PQ}$ where P(-4 , 9) and Q(5, -9). Calculate its coordinates.
c) Write the magnitude and gradient of $\overrightarrow{PQ}$ where P(6 , -2) and Q(4, -5). Calculate its coordinates.
d) Write the magnitude and gradient of $\overrightarrow{PQ}$ where P(-5 , 4) and Q(2, -8). Calculate its coordinates.
e) Write the magnitude and gradient of $\overrightarrow{PQ}$ where P(4 , -8) and Q(2, -1). Calculate its coordinates.
f) Write the magnitude and gradient of $\overrightarrow{PQ}$ where P(4 , -4) and Q(7, -1). Calculate its coordinates.
g) Write the magnitude and gradient of $\overrightarrow{PQ}$ where P(-8 , 6) and Q(9, -11). Calculate its coordinates.
h) Write the magnitude and gradient of $\overrightarrow{PQ}$ where P(-6 , 10) and Q(-4, 8). Calculate its coordinates.
i) Write the magnitude and gradient of $\overrightarrow{PQ}$ where P(12 , -9) and Q(-5, 17). Calculate its coordinates.
j) Write the magnitude and gradient of $\overrightarrow{PQ}$ where P(8, -8) and Q(9, -15). Calculate its coordinates.
k) Write the magnitude and gradient of $\overrightarrow{PQ}$ where P(16 , 18) and Q(9, -10). Calculate its coordinates.
l) Write the magnitude and gradient of $\overrightarrow{PQ}$ where P(6 , 10) and Q (5, -4). Calculate its coordinates.

6) Check if the following sets of points are aligned:
a) A(–1,3) B(-2.5,0.5) C(–4, –2)
b) A(1,0) B(–3, –2) C(5,2)

7)

a) Solve $\vec{v}+\left|\vec{w}-2\vec{u}\right|$ where $\vec{v}=(1,9), \vec{w}=(-2,4), \vec{u}=(3,-8)$

b) Solve $\vec{v}+\left|\vec{w}-3\vec{u}\right|$ where $\vec{v}=(2,-5), \vec{w}=(-1,8), \vec{u}=(4,9)$

c) Solve $\vec{v}-\left|\vec{w}+2\vec{u}\right|$ where $\vec{v}=(9,-2), \vec{w}=(7,-1), \vec{u}=(3,5)$

d) Solve $\vec{v}-\left|2\vec{w}-3\vec{u}\right|$ where $\vec{v}=(5,7), \vec{w}=(1,4), \vec{u}=(3,-8)$

e) Solve $\vec{v}+2\left|3\vec{w}-\vec{u}\right|$ where $\vec{v}=(-1,-4), \vec{w}=(3,6), \vec{u}=(-2,8)$

f) Solve $\vec{v}+3\left|\vec{w}-2\vec{u}\right|$ where $\vec{v}=(8,-7), \vec{w}=(7,-4), \vec{u}=(1,2)$

g) Solve $\vec{v}-\left|\vec{w}-3\vec{u}\right|$ where $\vec{v}=(4,5), \vec{w}=(-2,9), \vec{u}=(1,-6)$

h) Solve $\vec{v}+2\left|\vec{w}-2\vec{u}\right|$ where $\vec{v}=(3,-7), \vec{w}=(6,-9), \vec{u}=(4,5)$

i) Solve $\vec{v}-\left|2\vec{w}-\vec{u}\right|$ where $\vec{v}=(4,-2), \vec{w}=(5,-7), \vec{u}=(1,9)$

j) Solve $\vec{v}-2\left|\vec{w}+\vec{u}\right|$ where $\vec{v}=(5,4), \vec{w}=(-5,8), \vec{u}=(2,7)$

k) Solve $\vec{v}+\left|2\vec{w}-\vec{u}\right|$ where $\vec{v}=(8,-3), \vec{w}=(2,9), \vec{u}=(1,-6)$

l) Solve $\vec{v}-\left|\vec{w}+2\vec{u}\right|$ where $\vec{v}=(2,-5), \vec{w}=(3,4), \vec{u}=(6,-7)$

8)

a) Calculate $\vec{w}\cdot\vec{u}$ where $\vec{w}=(-2,4), \vec{u}=(3,-8)$

b) Calculate $\vec{w}\cdot\vec{u}$ where $\vec{w}=(3,4), \vec{u}=(6,-7)$

c) Calculate $\vec{w}\cdot\vec{u}$ where $\vec{w}=(2,9), \vec{u}=(1,-6)$

d) Calculate $\vec{w}\cdot\vec{u}$ where $\vec{w}=(-5,8), \vec{u}=(2,7)$

e) Calculate $\vec{w}\cdot\vec{u}$ where $\vec{w}=(5,-7), \vec{u}=(1,9)$

f) Calculate $\vec{w}\cdot\vec{u}$ where $\vec{w}=(6,-9), \vec{u}=(4,5)$

g) Calculate $\vec{w}\cdot\vec{u}$ where $\vec{w}=(-2,9), \vec{u}=(1,-6)$

h) Calculate $\vec{w}\cdot\vec{u}$ where $\vec{w}=(7,-4), \vec{u}=(1,2)$

i) Calculate $\vec{w}\cdot\vec{u}$ where $\vec{w}=(3,6), \vec{u}=(-2,8)$

j) Calculate $\vec{w}\cdot\vec{u}$ where $\vec{w}=(1,4), \vec{u}=(3,-8)$

k) Calculate $\vec{w}\cdot\vec{u}$ where $\vec{w}=(7,-1), \vec{u}=(3,5)$

l) Calculate $\vec{w}\cdot\vec{u}$ where $\vec{w}=(-1,8), \vec{u}=(4,9)$

9)

a) Write the magnitude and the gradient of $\vec{v}(5,7)$. Is it parallel to $\vec{t}(10,-14)$? Why?

b) Write the magnitude and the gradient of $\vec{v}(-1,-4)$. Is it parallel to $\vec{t}(2,8)$? Why?

c) Write the magnitude and the gradient of $\vec{v}(8,-7)$. Is it perpendicular to $\vec{t}(14,16)$? Why?

d) Write the magnitude and the gradient of $\vec{v}(4,5)$. Is it perpendicular to $\vec{t}(3,4)$? Why?

e) Write the magnitude and the gradient of $\vec{v}(8,-3)$.

Which of the following are perpendicular to it? $\vec{w}\left(1,\frac{8}{3}\right), \vec{u}(3,8)$

f) Write the magnitude and the gradient of $\vec{v}(2,-5)$. Draw it with initial point P(4,4).

10)

a) Write the magnitude and the gradient of $\vec{v}(1,9)$. Draw that vector placing the initial point in Q(2 , – 1)

b) Write the magnitude and the gradient of $\vec{v}(2,-5)$. Think of a parallel vector and draw both of them.

c) Write the magnitude and the gradient of $\vec{v}(9,-2)$. Think of a perpendicular vector and draw both of them.

d) Write the magnitude and the gradient of $\vec{v}(3,-7)$. Think of a parallel vector and draw both of them.

e) Write the magnitude and the gradient of $\vec{v}(4,-2)$. Think of a perpendicular vector and draw both of them.

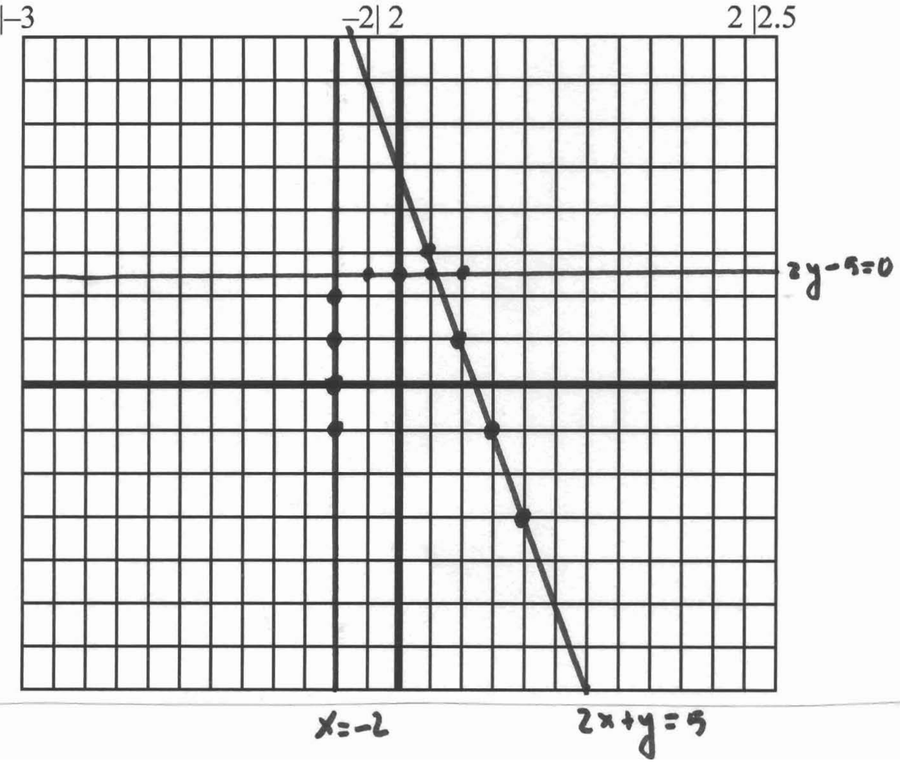
f) Write the magnitude and the gradient of $\vec{v}(5,4)$. Which of the following are parallel to it? $\vec{w}(2'5,2)$, $\vec{u}(10,6)$

Straight lines and linear functions

If we plot the graph of a linear function $y=mx+n$ we obtain a straight line. The y-intercept point is (0,n) and the **slope** (or **gradient**) is "m". The constant "n" is called the **ordinate.**

The equation may be expressed in the form $ax+by=c$ or any other equivalent expression.

Examples:	$2x+y=5$	$3x+6=0$	$2y-5=0$
	a linear function	this is not a function	a constant function
	$y=-2x+5$	$x=-2$	$y=2.5$
	x | y	x | y	x | y
	1 | 3	–2 | –1	–1 | 2.5
	2 | 1	–2 | 0	0 | 2.5
	3 | –1	–2 | 1	1 | 2.5
	4 | –3	–2 | 2	2 | 2.5

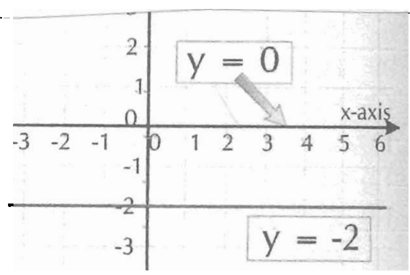


Special cases of straight lines

Horizontal line

An equation like y=c is satisfied by the coordinates of all the points in the horizontal line passing through (0,c)

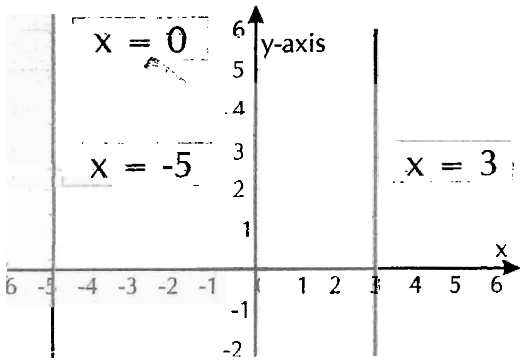
y = c is a horizontal line through "c" on the y-axis



Don't forget: the x-axis is also the line y = 0

Vertical line

An equation like x=c is satisfied by the coordinates of all the points in the vertical line passing through (c,0)



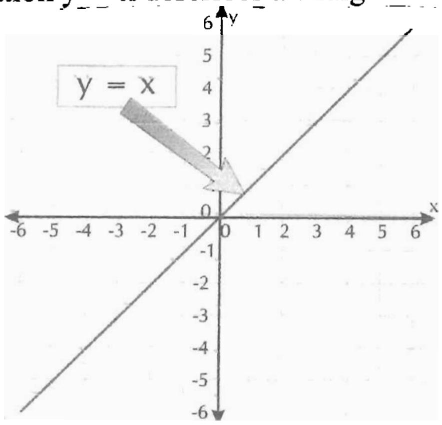
x = a is a vertical line through "a" on the x-axis

Don't forget: the y-axis is also the line x = 0

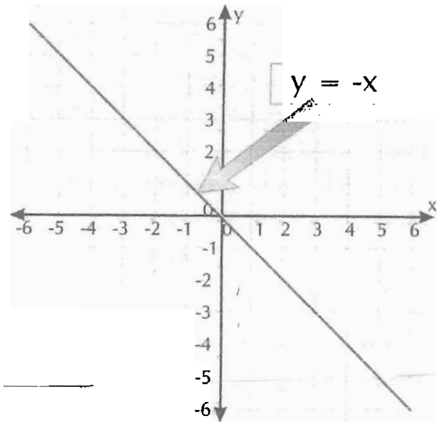
Main diagonals

The equation y= x describes a straight line through the origin (0,0) that bisects the first and third quadrant.

The equation y= –x describes a straight line through the origin (0,0) that bisects the second and fourth quadrant.



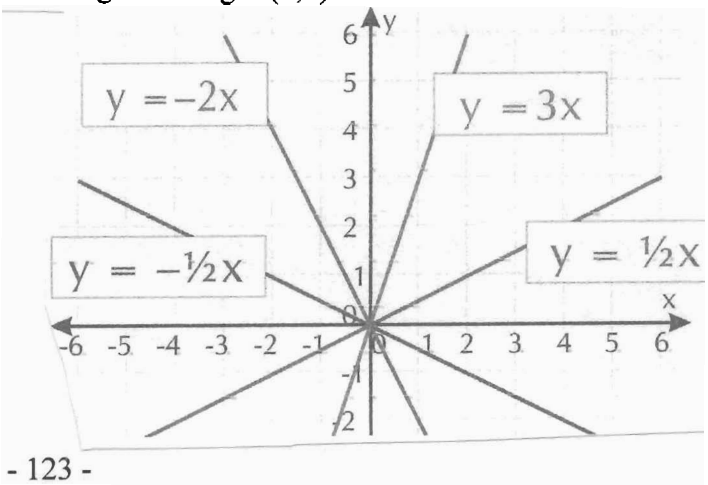
"y = x" is the main diagonal that goes UPHILL from left to right.



"y = -x" is the main diagonal that goes DOWNHILL from left to right.

Lines through the origin

An equation like y=mx describes a straight line through the origin (0,0)



The equations of the straight line

A **straight line** in the plane can be defined by **one of its points** $P(x_0, y_0)$ **and a direction vector** $\vec{v}(v_x, v_y)$ (there is only one line passing through a point with a certain slope).

Each point $Q(x, y)$ in the straight line has the following property: $\overrightarrow{PQ}$ is parallel to $\vec{v}$

Conversely, if a point $Q(x, y)$ satisfies that $\overrightarrow{PQ}$ is parallel to $\vec{v}$, then it is in the straight line.

So we have that:

$$\boxed{Q(x,y) \text{ is in the line}}$$

$$\Updownarrow$$

$$\boxed{\overrightarrow{PQ} \parallel \vec{v}}$$

$$\Updownarrow$$

$$\boxed{\overrightarrow{PQ} = \lambda \cdot \vec{v} \text{ for some } \lambda \in \Re}$$

$$\Updownarrow$$

$$\boxed{\begin{cases} x - x_0 = \lambda \cdot v_x \\ y - y_0 = \lambda \cdot v_y \end{cases} \text{ for some } \lambda \in \Re}$$

$$\Updownarrow$$

$$\boxed{\begin{cases} = x_0 + \cdot \\ y = y_0 + \lambda \cdot v_y \end{cases} \text{ for some } \lambda \in \Re}$$ the **parametric equations** of the straight line

$$\Updownarrow$$

$$\boxed{Q(x,y) = P(x_0, y_0) + \lambda \cdot \vec{v} \text{ for some } \lambda \in \Re}$$ the **vector equation** of the straight line

The equation of the straight line that passes through point $P(x_0, y_0)$ in the direction of vector $\vec{v}(v_x, v_y)$ can also have the following different forms:

$$\begin{cases} x = x_0 + \lambda \cdot v_x \\ y = y_0 + \lambda \cdot v_y \end{cases} \text{ for some } \lambda \in \Re \Leftrightarrow \begin{cases} \dfrac{x - x_0}{v_x} = \lambda \\ \dfrac{y - y_0}{v_y} = \lambda \end{cases} \text{ for some } \lambda \in \Re \Leftrightarrow \boxed{\frac{x - x_0}{v_x} = \frac{y - y_0}{v_y}}$$

$$\frac{x - x_0}{v_x} = \frac{y - y_0}{v_y} \Leftrightarrow \frac{v_y}{v_x}(x - x_0) = y - y_0 \Leftrightarrow \boxed{y - y_0 = m_{\vec{v}}(x - x_0)}$$ the **point-slope** form

$$\frac{x - x_0}{v_x} = \frac{y - y_0}{v_y} \Leftrightarrow v_y(x - x_0) = v_x(y - y_0) \Leftrightarrow v_y x - v_y x_0 = v_x y - v_x y_0 \Leftrightarrow v_y x - v_x y + v_x y_0 - v_y x_0 = 0 \Leftrightarrow$$

$$\boxed{\begin{array}{c} Ax + By + C = 0 \\ \textit{being} \\ A = v_y, B = -v_x, C = v_x y_0 - v_y x_0 \end{array}}$$ the **general** form

or else

$$\boxed{\begin{array}{c} Ax + By = C \\ \textit{being} \\ A = v_y, B = -v_x, C = v_y x_0 - v_x y_0 \end{array}}$$ the **standard** form

$$y - y_0 = m_{\vec{v}}(x - x_0) \Leftrightarrow y = m_{\vec{v}} x - m_{\vec{v}} x_0 + y_0 \Leftrightarrow \boxed{\begin{array}{c} y = mx + n \\ \textit{being} \\ n = -m_{\vec{v}} x_0 + y_0, m = m_{\vec{v}} \end{array}}$$ the **slope-intercept** form

Example: the forms of equation of the straight line that passes through $P(4,-1)$ in the direction of $\vec{v}(-2,3)$

$(x,y)=(4,-1)+\lambda(-2,3),\ \lambda\in\Re$ the **vector equation**

$$\left.\begin{array}{l} x=4+\lambda\cdot(-2) \\ y=-1+\lambda\cdot 3 \end{array}\right\},\lambda\in\Re \qquad \left.\begin{array}{l} x=4-2\lambda \\ y=-1+3\lambda \end{array}\right\},\ \lambda\in\Re$$ the **parametric equations**

$$\frac{x-4}{-2}=\frac{y-(-1)}{3} \qquad \frac{x-4}{-2}=\frac{y+1}{3}$$

$$\frac{3}{-2}(x-4)=y+1 \qquad y+1=\frac{-3}{2}(x-4)$$ the **point-slope** equation

$$3(x-4)=-2(y+1) \qquad 3x-12=-2y-2 \qquad 3x+2y-10=0$$ the **general** equation

$3x+2y=10$ the **standard** equation

$$y+1=\frac{-3}{2}(x-4) \qquad y=\frac{-3}{2}(x-4)-1 \qquad y=\frac{-3}{2}x+\frac{12}{2}-1 \qquad y=\frac{-3}{2}x+5$$ the **slope-intercept** equation

Slope or gradient of a straight line

The slope or gradient of a line is that of its direction vector: $m=m_{\vec{v}}=\dfrac{v_y}{v_x}$

When we have the general equation or the equation in standard form the slope is: $m=-\dfrac{A}{B}$

The gradient of a line joining the points $P(x_1,y_1)$ and $Q(x_2,y_2)$ is: $m=\dfrac{y_2-y_1}{x_2-x_1}$

Straight lines that are **parallel** must have the same gradient.
When two straight lines are **perpendicular,** the product of their gradients is -1.

Note: given the equation in general or standard form, the vector $\vec{u}(A,B)$ is perpendicular to the line.

Examples:

- the line with gradient 4 crossing the y-axis at -5 has equation $y=4x-5$.

- the gradient of the line joining $(2,-5)$ and $(-1,4)$ is $\dfrac{4-(-5)}{-1-2}=\dfrac{9}{-3}=-3$.

- Find the equation of the straight line perpendicular to the line $4x+3y=12$ and passing through the point (2,5).

Rearrange the equation into gradient–intercept form: $y=-\frac{4}{3}x+4$.
If the gradient of the required line is m then: $m\times -\frac{4}{3}=-1 \Rightarrow m=\frac{3}{4}$.
Using the form $y-y_1=m(x-x_1)$ gives: $y-5=\frac{3}{4}(x-2)$.

- Find the equation of the straight line parallel to $y=3x-5$ and passing through the point (4, 2).

The line must have gradient 3 and so it can be written in the form $y=3x+c$.
Substituting $x=4$ and $y=2$ gives $2=3\times 4+c \Rightarrow c=-10$.
The required equation is $y=3x-10$.

- Find the equation of the straight line passing through the points (−1,5) and (3,−2).

One approach is to use the form $y=mx+c$ to produce a pair of simultaneous equations:

Substituting $x=-1$ and $y=5$ gives $\quad 5=-m+c \quad$ (1)
Substituting $x=3$ and $y=-2$ gives $\quad -2=3m+c \quad$ (2)

(2) − (1) gives: $-7=4m \Rightarrow m=\frac{-7}{4}$.

Substituting for m in (1) gives: $5=\frac{7}{4}+c \Rightarrow c=\frac{13}{4}$

The equation of the line is $y=\frac{-7}{4}x+\frac{13}{4}$. This is the same as $4y+7x=13$.

An alternative approach is to find the gradient directly and then use the form $y-y_1=m(x-x_1)$.

Taking (−1, 5) as (x_1, y_1) and (3, −2) as (x_2, y_2),
$m=\frac{-2-5}{3-(-1)}=\frac{-7}{4}=\frac{7}{4}$.

The equation of the line is $y-5=-\frac{7}{4}(x+1)$.

EXERCISES

11) Write all the equations of the line passing through point P in the direction of vector $\vec{v}$:
   a) $P(-7,2), \vec{v}(2,1)$
   b) $P(4,-3), \vec{v}(-5,4)$
   c) $P(5,1), \vec{v}(-1,-3)$

12) Tell a direction vector and the gradient of the line passing through points A and B. Then write one equation of the line:
   a) $A(-1,0), B(0,3)$
   b) $A(0,-2), B(5,-2)$
   c) $A(-2,3), B(4,-1)$

13) Write an equation of the line that:
   a) Passes through point $(1,3)$ and has direction vector $\vec{v}(2,-1)$
   b) Passes through point $(-2,1)$ and has direction vector $\vec{v}(-1,-3)$
   c) Passes through point $(3,-2)$ and has direction vector $\vec{v}(2,0)$

14) Calculate the equation of the following lines:
   a) A line parallel to $y=-2x+3$ passing through $(4,5)$
   b) A line parallel to $2x-4y+3=0$ passing through $(4,0)$
   c) A line parallel to $3x+2y-6=0$ passing through $(0,-3)$

15) Calculate the equation of the following lines:
   a) A line passing through $(-4,2)$ with slope $m=\frac{1}{2}$
   b) A line passing through $(1,3)$ with slope $m=-2$
   c) A line passing through $(5,-1)$ with slope $m=0$

16) Write one equation of the line passing through point P that is perpendicular to vector $\vec{v}$:
   a) $P(-7,2), \vec{v}(2,1)$
   b) $P(4,-3), \vec{v}(-5,4)$
   c) $P(5,1), \vec{v}(-1,-3)$

17) Write the equation of a line perpendicular to $3x-y+6=0$ containing point $(-3,0)$

18) Calculate the gradient and a direction vector of a line perpendicular to the one that passes through points $A(3,1) and B(-5,-1)$

19) The equation of a straight line is $y=2x-7$.
   (a) Write down the equation of a parallel line crossing the y-axis at 5.
   (b) Find the equation of a parallel line passing through the point (4,9).

20) Find a vector equation of the line through the points $P(3,1)$ and $Q(-2,5)$

Distance between two points

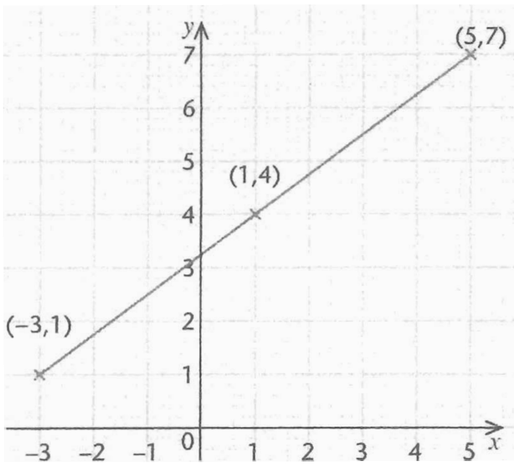
The distance between the points A(x_1, y_1) and B(x_2, y_2) is given by $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$. The result is based on Pythagoras' theorem.

For example, the distance between (−3,4) and (2,−8) is given by $\sqrt{(-3-2)^2 + (4-(-8))^2} = \sqrt{5^2 + 12^2} = 13$ units.

Middle point of a segment

If P has coordinates (x_1, y_1) and Q has coordinates (x_2, y_2) then the mid-point of PQ has coordinates $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

For example, the mid-point of the line joining (−3,1) and (5,7) is $\left(\frac{-3+5}{2}, \frac{1+7}{2}\right) = (1,4)$.

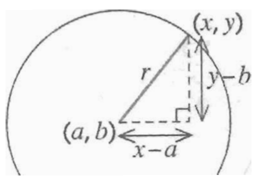


Equation of a circle

The equation of a circle with centre (a, b) and radius r is $(x-a)^2 + (y-b)^2 = r^2$.

This result is based on Pythagoras' theorem.

$(x-a)^2 + (y-b)^2 = r^2$



An alternative form of the equation of a circle is $x^2 + y^2 + 2gx + 2fy + c = 0$.
The centre of the circle is $(-g, -f)$ and the radius is $\sqrt{g^2 + f^2 - c}$.

For example, the equation of a circle with centre (4,0) and radius 5 units is $(x-4)^2 + y^2 = 25$.

For example, the circle with equation $x^2 + y^2 + 6x - 10y + 18 = 0$ has centre (−3,5) and radius $\sqrt{(-3)^2 + 5^2 - 18} = \sqrt{16} = 4$ units.

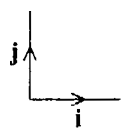
You can also show this by completing the square on the x terms and on the y terms as follows:

$$x^2 + 6x + y^2 - 10y + 18 = 0$$
$$(x+3)^2 - 9 + (y-5)^2 - 25 + 18 = 0$$
$$(x+3)^2 + (y-5)^2 = 16$$

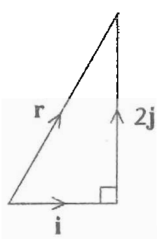
The centre is at (−3, 5) and the radius is 4.

Component form of a vector

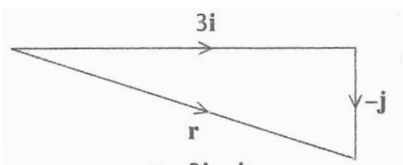
When working in two dimensions, it is often very useful to express a vector in terms of two special vectors **i** and **j**. These are **unit vectors** at right-angles to each other.
A vector **r** written as $\mathbf{r} = a\mathbf{i} + b\mathbf{j}$ is said to have **components** a**i** and b**j**.



Examples



r = i + 2j

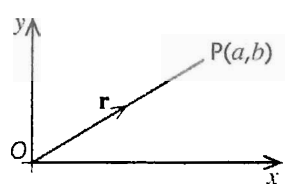


r = 3i − j

For work involving the Cartesian coordinate system, **i** and **j** are taken to be in the positive directions of the x- and y-axes respectively.

The **position vector** of a point P is the vector $\overrightarrow{OP}$ where O is the origin.

If P has coordinates (a, b) then its position vector is given by $\mathbf{r} = a\mathbf{i} + b\mathbf{j}$.



In three dimensions, a point with coordinates (a, b, c) would have position vector $\mathbf{r} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$.

EXERCISES

21) Find the equation of the straight line joining the points (–2,–9) and (1,3).

22) The equation of a straight line is $3x + 2y = 11$
(a) Find the equation of a parallel line passing through the point (5,4).
(b) Find the equation of a perpendicular line passing through the point (–2,6).

23) Find the coordinates of the mid-point of (–9,3) and (–5,1).

24) Calculate the distance between P and Q:
a) $P(3,5)$ and $Q(3,-7)$
b) $P(-8,3)$ and $Q(-6,1)$
c) $P(0,-3)$ and $Q(-5,1)$
d) $P(-3,0)$ and $Q(15,0)$

25) a) Find the middle point of the segment with end-points $A(-2,0) and B(6,4)$
b) Check that the distance between the middle point and any of the end-points is the same.

26) P is the point (–1,–4) and Q is the point (5,–1). Find the equation of the line perpendicular to PQ and passing through its mid-point.

27) A is the point (4,–7) and B is the point (–2,1).
Find the distance AB.

28) Write the equation of the circle with centre "C" and radius "r" units:
a) $C(4,-3), r = 3$
b) $C(0,5), r = 6$
c) $C(6,0), r = 2$
d) $C(0,0), r = 5$

29) (a) Find the equation of a circle with centre (–3,8) and radius 5 units.
(b) A circle has equation $x^2 + y^2 - 12x + 8y + 43 = 0$.
Find its centre and radius.

30) Say the centre and the radius of the following circles:
a) $(x-2)^2 + (y+3)^2 = 16$
b) $(x+1)^2 + y^2 = 81$
c) $x^2 + y^2 = 10$