Random events and probability.

Random events happen by chance. **Probability** is a measure of how likely they are. In a trial (or experiment) the things that can happen are called **outcomes**. Events are groups of one or more **outcomes**.

Example: "time to eat my dinner" can be an experiment
"63 seconds" is an outcome
"less than a minute" is an event.

Laplace's law

When all outcomes are **equally likely** you can work out the probability of an event by counting the outcomes:

$$P(\text{event}) = \frac{\text{Number of outcomes where event happens}}{\text{Total number of possible outcomes}}$$

Example: "you have got a bag with 15 balls in (5 red, 6blue and 4 yellow); you take a ball out without looking"
15 possible outcomes so

$$P(red\text{–}ball) = \frac{5}{15} = \frac{1}{3} \qquad P(blue\text{–}ball) = \frac{6}{15} = \frac{2}{5} \qquad P(yellow\text{–}ball) = \frac{4}{15}$$

$$P(red-or-yellow-ball) = \frac{9}{15} = \frac{3}{5}$$

Sample space

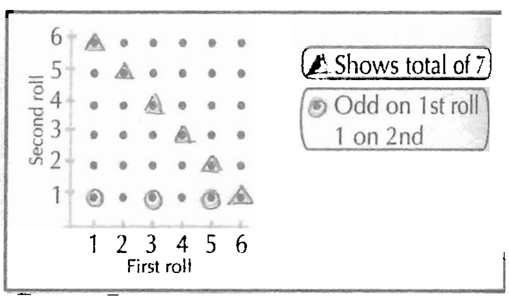
The **sample space** S is the set of all possible outcomes. It must be considered in the process of calculating probabilities of events.

Example: The classic probability machine is a dice. If you roll it twice, you can record all the possible outcomes in a 6 × 6 table (a possible diagram of the sample space).

There are 36 outcomes in total. You can find probabilities by counting the ones you're interested in (and using the above formula). For example:

(i) The probability of an odd number and then a '1'. There are 3 outcomes that make up this event, so the probability is: $\frac{3}{36} = \frac{1}{12}$

(ii) The probability of the total being 7. There are 6 outcomes that correspond to this event, giving a probability of: $\frac{6}{36} = \frac{1}{6}$



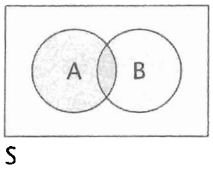
Set algebra and Venn diagrams

Venn diagrams provide a useful way to represent information about **sets** of objects.

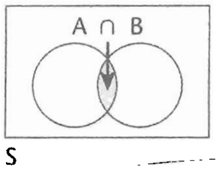
The reason for mentioning them here is that the diagrams, and the notation used to express results, have a direct interpretation and application in probability theory.

In each diagram, the rectangle represents the set S of all objects under consideration.

The circles represent particular sets A and B of objects within the set S.

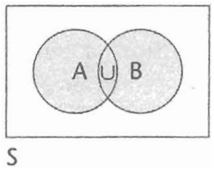


A ∩ B represents the set of those objects that belong to *both* A *and* B.



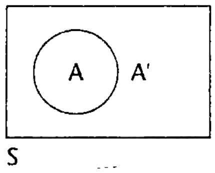
A∩B is read as "A intersection B"

A ∪ B represents the set of all objects that belong to *either* A *or* B *or both*.



A∪B is read as "A union B"

A′ represents the set of objects in S that do not belong to A.



A' is read as "complement of A"

Set algebra and probability theory

To make use of these ideas in probability theory:

- The objects are interpreted as **outcomes** for a particular situation.
- The set S is the **sample space** of all possible outcomes for the situation.
- The sets A and B are **events** defined by a particular choice of outcomes.
- P(A) for **example**, represents the probability that the event A occurs.
- P(A ∪ B) represents the probability that *either* A *or* B *or both* occurs.
- P(A ∩ B) represents the probability that *both* A *and* B occur.
- P(A′) represents the probability that A does *not* occur.

The **addition rule** states that

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

A and B are described as **mutually exclusive events** if they have no outcomes in common. These are events that cannot both occur at the same time.

In this case $P(A \cap B) = 0$ and the addition law becomes

$$P(A \cup B) = P(A) + P(B).$$

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$P(A \cap B) = 0$

The events A and A′ are mutually exclusive for any event A.
It follows that $P(A \cup A') = P(A) + P(A')$.
Since one of the events A or A′ must occur, $P(A \cup A') = 1$ so $P(A) + P(A') = 1$.
This is usually written as $P(A') = 1 - P(A)$.

The **multiplication rule** states that:

$$P(A \cap B) = P(A \mid B) \times P(B),$$

where $P(A \mid B)$ means the probability that A occurs *given that* B has *already* occurred.

$P(A \mid B)$ is described as a **conditional probability**, i.e. it represents the probability of A *conditional* upon B having occurred. Re-arranging the multiplication rule gives

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}.$$

If A and B are **independent events** then the probability that either event occurs is not affected by whether the other event has already occurred.

In this case, $P(A \mid B) = P(A)$ and the multiplication rule becomes

$$P(A \cap B) = P(A) \times P(B).$$

A **tree diagram** is a useful way to represent the probabilities of combined events. Each path through the diagram corresponds to a particular sequence of events and the multiplication rule is used to find its probability. When more than one path satisfies the conditions of a problem, these probabilities are added.

Examples:

- The events A and B are independent. P(A) = 0.3 and P(B) = 0.6.
  Find (a) $P(A \cup B)$ (b) $P(A' \cap B)$.

  (a) (i) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

  Since A and B are independent, $P(A \cap B) = P(A) \times P(B)$.

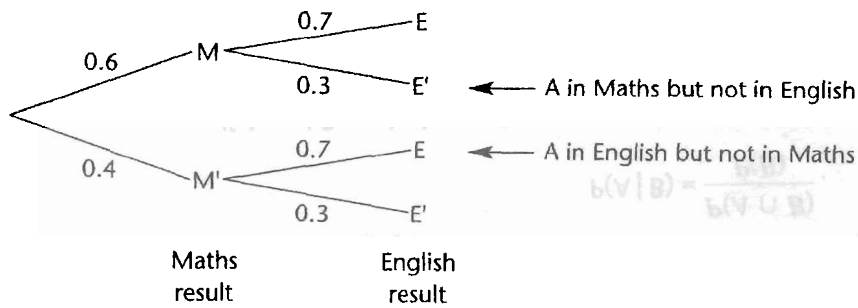
  So $P(A \cup B) = 0.3 + 0.6 - 0.3 \times 0.6 = 0.72$.

  (b) A and B are independent $\Rightarrow$ A' and B are independent.

  So $P(A' \cap B) = P(A') \times P(B) = (1 - 0.3) \times 0.6 = 0.42$.

- The probability that Chris gets grade A for Maths is 0.6 and the corresponding probability for English is 0.7. The events are independent.

  (a) Calculate the probability that Chris gets just one A in the two subjects.
  (b) Given that Chris gets just one grade A, find the probability that it is for Maths.



  (a) $P(\text{just one A}) = 0.6 \times 0.3 + 0.4 \times 0.7$
  $= 0.18 + 0.28$
  $= 0.46$

EXERCISES

1) When a dice is rolled once the probability of turning a 6 is $\frac{\square}{\square}$.

2) This bag contains 3 red and 6 blue cubes.

(a) The probability of drawing a red cube is $\frac{\square}{\square} = \frac{\square}{\square}$.

(b) The probability of drawing a blue cube is $\frac{\square}{\square} = \frac{\square}{\square}$.

3) Imagine a bag containing 3 yellow, 6 green and 6 red cubes.
The probability of drawing

(a) a yellow cube is $\frac{\square}{\square}$

(b) a green cube is $\frac{\square}{\square}$

(d) a yellow or green is $\frac{\square}{\square}$

(e) a yellow or red is $\frac{\square}{\square}$

(c) a red cube is $\frac{\square}{\square}$

(f) a green or red is $\frac{\square}{\square}$

4) The bag now contains 5 red, 5 blue and 5 black cubes.

(a) The probability of drawing a blue cube is $\frac{\square}{\square}$.

(b) What is the probability of drawing a red cube? $\frac{\square}{\square}$.

(c) The probability of drawing a red, a blue or a black cube is ________

(d) The probability of drawing a pink cube is ________

(e) There is an equal ________ of drawing a red, a blue or a black cube.

5) A bag contains 10 balls with numbers 1 to 10 written in them. Consider the experiment "taking a ball out of the bag".

a) Describe the sample space.

b) If A="get a prime number" and B="get a multiple of 3" write the events A, B, A', B', $A \cup B$, $A \cap B$, $A \cup A'$, $A \cap A'$

6) The probability that the event B occurs is 0.7.
The probability that events A and B both occur is 0.4.
What is the probability that A occurs given that B had already occurred?

7) It is given that $P(A)=0.4$, $P(B)=0.7$ and $P(A \cap B)=0.2$. Find:
   a) $P(A \cup B)$
   b) $P(B|A)$
   c) $P(A|B)$

8) A bag contains 3 red counters and 7 green counters.
Two counters are taken at random from the bag, without replacement.
   a) By drawing a tree diagram, or otherwise, find the probability that the counters are different colours.
   b) Given that the counters are different colours, find the probability that the first counter picked is red.