Combinatorics

Combinatorics is a branch of mathematics that studies collections of objects that satisfy specified criteria. In particular, it is concerned with "counting" the number of arrangements or selections of objects.

Example: What is the number of possible orderings of a deck of 52 playing cards? The answer is 52! (fifty-two factorial): $52 \cdot 51 \cdot 50 \cdot 49 \cdot \ldots \cdot 3 \cdot 2 \cdot 1$

the number of possible arrangements

Description	No of arrangements
n different objects in a straight line.	$n!$
n objects in a straight line, where r objects are the same and the rest are different.	$\frac{n!}{r!}$
n objects in a straight line, where r objects of one kind are the same, q objects of another kind are the same and the rest are different.	$\frac{n!}{q!r!}$

Permutations

Permutation is the rearrangement of objects or symbols an ordered list without repetitions, perhaps missing some elements. Each unique ordering is called a permutation.

The **number of permutations of a sequence** is: ${}^{n}P_{r} = \frac{n!}{(n-r)!}$

r is the size of each permutation,
n is the size of the sequence from which elements are permuted, and
! is the factorial operator.

Note: an easier way to compute this is to take the first r numbers of n factorial.

Examples:
- we have a total of 10 elements, the integers {1, 2, ..., 10}
  a permutation of three elements from this set is for example (5, 3, 4)
  to find out how many unique sequences, such as the one previously we consider (n = 10 and r = 3) and calculate

$$ {}^{10}P_{3} = \frac{10!}{7!} = 10 \cdot 9 \cdot 8 = 720 $$

- How many different permutations are there of the numbers 5, 7 and 9?

The way to do these is to think: 'How many choices do I have for the first number?',
'How many choices do I then have for the second number?' and so on.
You can put either 5, 7 or 9 in the first position, so you have 3 choices — I'll choose 7.
Then you can put either 5 or 9 in the second position, so you have 2 choices — I'll choose 9.
Then I have only 1 choice for what goes in the third position — I have to choose 5.
So there are 3 × 2 × 1 = 6 permutations of 5, 7 and 9.

- How many permutations of 5 letters can you make from the letters of the alphabet (if you don't use any letter twice)?

You have 26 choices for the first letter, 25 for the second, then 24, 23 and 22 for the third, fourth and fifth.
This gives a total of 26 × 25 × 24 × 23 × 22 = 7 893 600 permutations. This is $\frac{26!}{21!}$.

If some of the objects or symbols are equal and undistinguishable, the **number of permutations** of all of them is: $^{n}P_{r_1,r_2,r_3,...} = \frac{n!}{r_1!\cdot r_2!\cdot r_3!\cdot ...}$

n is the size of the sequence from which elements are permuted, and
! is the factorial operator.
$r_1, r_2, r_3...$ there are r_1 equal objects, r_2 equal objects, r_3 equal objects, ...

Examples: - we have a total of 10 balls,
{red, red, green, green, green, blue, black, brown, yellow, white}
a permutation of the elements from this set is for example
(red, green, red, green, blue, black, brown, yellow, white, green)
to find out how many unique sequences, such as the one previously we consider ($n = 10$ and $r_1 = 2$, $r_2=3$) and calculate

$$\frac{10!}{2!\cdot 3!} = \frac{10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{2\cdot 1\cdot 3\cdot 2\cdot 1} = 302400$$

- How many different arrangements of the letters in the word MISSISSIPPI are there?

There are 11 letters in total — including 4 S's, 4 I's and 2 P's.

So the number of arrangements is: $\frac{11!}{4!\times 4!\times 2!} = 34650$

Permutation with repetition

When order matters and an object can be chosen more than once then the **number of permutations with repetition** is n^r

n is the number of objects from which you can choose
r is the number to be chosen.

Example : - you have the letters A, B, C, and D and wish to calculate the number of ways to arrange them in three letter patterns (trigrams)
order matters, an object can be chosen more than once, n=4, r=3
so the number of trigrams is $4^3 = 64$ ways.

Combination without repetition

Selections where the order does not matter and each object can be chosen only once are called **combinations**. The **number of combinations** is the binomial coefficient $^{n}C_r = \frac{n!}{(n-r)!\,r!} = \binom{n}{r}$

r is the size of each combination,
n is the size of the sequence from which elements are chosen, and
! is the factorial operator.

Examples: ■ you have ten numbers and wish to choose 5

you would have $^{10}C_5 = \frac{10!}{5!5!} = \frac{10\cdot 9\cdot 8\cdot 7\cdot 6}{5\cdot 4\cdot 3\cdot 2\cdot 1} = 252$ ways to choose.

■ How many ways are there to choose a team of 11 players from a squad of 16?

Easy — just use the formula: $\binom{16}{11} = \frac{16!}{5!\times 11!} = 4368$

Notice that $\binom{16}{11} = \binom{16}{5}$.

■ How many ways are there to choose 6 Lotto numbers from a possible 49?

Use the formula again: $\binom{49}{6} = \frac{49!}{6!\times 43!} = 13\,983\,816$

Cancel the 43! with the rest of the 49! to get

$\frac{49!}{6!43!} = \frac{49\times 48\times 47\times 46\times 45\times 44}{6\times 5\times 4\times 3\times 2\times 1}$

Combination with repetition

When the order does not matter and an object can be chosen more than once, we make **combinations with repetition**. The **number of combinations with repetition** is

$$^{n}CR_r = \frac{(n+r-1)!}{(n-1)!\,r!} = \binom{n+r-1}{r} = \binom{n+r-1}{n-1}$$

n is the number of objects from which you can choose
r is the number to be chosen.

Example: you have three types of donuts (n) on a menu to choose from
you want ten donuts (r)
there are $^{3}CR_{10} = \frac{12!}{2!10!} = \frac{12\cdot 11}{2\cdot 1} = 66$ ways to choose

EXERCISES

1) Find the number of arrangements of the letters STATISTICS.

2) Find the number of combinations of seven objects chosen from ten.

3) There's 1 white ball, 1 red ball and 1 black ball in an urn.
We pick them one by one. Write all the possible ways of doing it.

4) Two friends play tennis and consider as winner the first one that wins two sets.
In how many ways can the play come out? Write them down.

5) Make all the possible 4-figure numbers with 1 and 2.

6) Suppose you wish to make two-coloured pencils and you have
red, blue, black, green and violet colours.
How many models of pencil can be made? Write down all of them.

7) Which numbers of two different figures can be formed with 1, 2, 3, 4 and 5?

8) Describe the pieces of a domino made with numbers 1, 2, 3, 4 and 5?

9) If you have got three pair of trousers and four t-shirts,
in how many ways can you be dressed? Describe all of them.

10) The following are solutions for the questions below 8^2 $8!$ ${}^8P_2$ ${}^8C_2$
Match each question with its solution:
a) 8 letters words (even senseless) that can be formed with the letters of
CAPELONI
b) Pairs that can be formed to play a chess competition amongst 8 people.
c) Two-figure numbers that can be formed with digits 1, 2, 3, 4, 5, 6, 7 and 8.
d) Ways of giving the first and second awards of a literary contest
in which 8 people take part.

11) Calculate: a) 4^3 b) ${}^7P_3$ c) ${}^7P_7$ d) ${}^6C_4$
e) ${}^9P_5$ f) $\dfrac{10!}{8!}$ g) ${}^{10}C_8$

12) Calculate: a) ${}^5V_2 - {}^5C_3$ b) $\dfrac{6^2}{{}^4C_2}$ c) $\dfrac{{}^4V_4}{{}^4V_3}$
d) $\dfrac{5!}{3!}$ e) $\dfrac{20!}{19!}$ f) $\dfrac{{}^{12}V_{12}}{{}^9V_9}$

13) a) How many permutations are there of the eight letters a, c, f, g, i, t, x, w?
b) Of the permutations in part (a) how many start with the letter t?
c) How many start with the letter g and end with the letter c?

14) In how many ways can the symbols a, b, c, d, e, e, e, e, e be arranged so that no e is adjacent to another e?

15) a) In how many ways can 7 people be arranged around a circular table?
b) If two of the people insist on sitting next to each other, how many arrangements are possible?

16) How many different paths in the xy-plane are there from (0,0) to (7,7) if a path proceeds one step at a time by either going one unit to the right or one unit up? How many paths are there from (2,7) to (9,14)?

17) A computer science professor has seven different programming books on a bookshelf, three of them dealing with C++ and the other four with Prolog. In how many ways can the books be arranged on the shelf if
a) there are no restrictions,
b) if the languages must alternate,
c) if all the C++ books must be next to each other, and
d) if all the C++ books must be next to each other and all the Prolog books must be next to each other?

18) List all the combinations of size 3 possible from the set {m, r, a, f, t}.

19) A committee of 12 is to be formed from 10 men and 8 women candidates. In how many ways can the committee be composed if
a) there are no restrictions,
b) there must be 6 men and 6 women,
c) there must be an even number of women,
d) there must be more women than men?

20) How many 8-bit bytes contain
a) exactly two 1's
b) exactly four 1's
c) exactly six 1's
d) at least six 1's

21) Imagine a primitive operating system that allows filenames of from one to eight characters where each character may be a letter, a digit, or one of 15 other symbols such as underscore, etc. Since this imaginary system is primitive, assume it does not distinguish between lower and upper case letters so there is a total of 26 + 10 + 15 = 51 possible characters. In addition, a filename may contain an extension, ie one to three alphanumeric characters after a full stop. Thus filenames could be EXAMPLE.DOC, PROG1.C, A13_D2, Table.P82, Results, ….
a) How many filenames use only the 36 alphanumeric characters and no extension
b) How many of the filenames in (a) start with AA?
c) How many filenames use extensions of exactly three alphanumeric characters

Binomial expansion

An expression which has two terms, such as $a+b$ is called a **binomial**. The expansion of something of the form $(a+b)^n$ is called a **binomial expansion**. When n is a positive integer:

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \binom{n}{3}a^{n-3}b^3 + \ldots + b^n$$

- Starting with a^n, the power of a is reduced by 1 each time until the last term, b^n, which is the same as a^0b^n.
- The power of b is increased by 1 each time. Notice that the first term, a^n, is the same as a^nb^0 and that the powers of a and b always add up to n in each term.
- It's useful to recognise that the term involving b^r takes the form $\binom{n}{r}a^{n-r}b^r$.
- For positive integer values $\binom{n}{r}$ has the same value as ${}^nC_r$ and you may find that your calculator will work this out for you.

Example Expand $(1+3x)^{10}$ in ascending powers of x up to and including the fourth term.

$$\begin{aligned}(1+3x)^{10} &= 1 + \binom{10}{1}(3x) + \binom{10}{2}(3x)^2 + \binom{10}{3}(3x)^3 + \ldots \\ &= 1 + 10(3x) + 45(9x^2) + 120(27x^3) + \ldots \\ &= 1 + 30x + 405x^2 + 3240x^3 + \ldots.\end{aligned}$$

Example Find the coefficient of the x^7 term in the expansion of $(3-2x)^{15}$

The term involving x^7 is given by $\binom{15}{7}(3)^8(-2x)^7$

$$= 6435 \times 6561 \times -128x^7,$$

and so the required coefficient is $-5\ 404\ 164\ 480$.

Pascal's triangle

The coefficients in the expansion of $(a+b)^n$ also follow the pattern given by the row of **Pascal's triangle** that starts with 1 n ...

```
                1
              1   1
            1   2   1
          1   3   3   1
        1   4   6   4   1
      1   5  10  10   5   1
```

Pascal's triangle: each row starts and ends with 1. Every other value is found by adding the pair of numbers immediately above it in the pattern.

- When n is small and the full binomial expansion is required, the simplest way to find the coefficients is to use Pascal's triangle. For larger values of n it may be simpler to use the formula, particularly if only some of the terms are required.

Examples:

- $(1+x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4$.
- $(1-2x)^4 = 1 + 4(-2x) + 6(-2x)^2 + 4(-2x)^3 + (-2x)^4$
  $= 1 - 8x + 24x^2 - 32x^3 + 16x^4$.

EXERCISES

22) An exnagram ("exclusive anagram") is a rearrangement of the characters making up a word such that in each character position there is a change of character. Thus EWIN is an exnagram of WINE but WEIN is not. The word BEER has only two exnagrams, EBRE and ERBE. The word SEE has 2 no exnagrams (in any rearrangement at least one E must occupy a place originally occupied by an E).
a) How many exnagrams are there of the word RADAR?
b) How many exnagrams are there of the word ANAGRAM?

23) A message of 12 different symbols is to be transmitted through a communications channel. In addition, the transmitter will also send 45 blank characters between the symbols, with at least three blanks between each pair of consecutive symbols. In how many ways can the transmitter send such a message?

24) In how many ways can we distribute eight identical balls into three containers
a) so that no container is left empty,
b) so that the third container contains an odd number of balls, and
c) so that no container is left empty and the third container contains an even number of balls?

25) Jack has 7 kinds of flowers and wishes to give Jill a bouquet with 3 of them. In how many ways can this be done?

26) Expand $(k + l)^3$ using Pascal's triangle

27) How many different 5-card hands can be dealt from a standard 52-card deck?

28) What is the probability that you win the Cash5 game, which involves selecting 5 numbers out of 34, and you have to match all 5 of them to win?

29) Bob goes to the ice cream shop to buy Alice and himself sundaes.
When he gets there, he realizes that he forgot what sort of sundae Alice wanted!
All he remembers is that she wanted 3 different flavours of ice cream (the shop offers 14) and 2 different kinds of topping (the shop offers 5).
Now, he knows it is important to Alice that he gets the right sundae, so he decides to get one of each possible type. How many sundaes does poor Bob have to buy?

30) How many 4-digit numbers are there whose digits appear in strictly increasing order?

31) How many different sequences of "H" and "T" are there which have exactly 3 H's and 7 T's? (Think of these as coin tosses.)

32) a) What is the coefficient of a^4b^3 in the expansion of $(a + b)^7$?
b) What is the coefficient of a^4b^3 in the expansion of $(2a + 3b)^7$?