MATHEMATICS

YEAR 4

FIRST TERM

Variable expressions

A **variable expression** consists of some numbers and letters joined by operations. The letters are called **variables** and they represent unknown quantities.

The **numerical value** of a variable expression is obtained by substituting actual numerical values for variables (that is "plugging in" numbers for the variables).

Examples: Evaluate $9y + 5$ when $y = 2$ $9\cdot 2 + 5 = 23$ the numerical value is 23

Evaluate $4xz + \frac{12}{x}$ when $x = 3$ and $z = -2$ $4\cdot 3\cdot(-2)+\frac{12}{3}=-24+4=-20$ it is -20.

Variable expressions are written for calculations involving quantities that can vary. **Algebra** is a language that uses variable expressions; the sentences of a problem must be transtaled into this language.

Examples:
A number.. x
The double of a number........................ $2x$
Three added to the square of a number......... $3 + x^2$
Half part of a number........................ $\frac{x}{2}$

Equations, identities, solutions

An **equation** (or **conditional equation**) is a mathematical statement that two variable expressions are equal. It consists of two expressions joined by the equals sign "=".

The values that the variables have when the equation is true are called **the solutions of the equation** (or **roots**).

An **identity** (**identical equation** or **true equation**) is an equality that states a fact. It is an equation which is true for all values of the variables.

Examples: $2 + x = 5$ $x^2 = 81$ $3^x = 243$ $2x = 6$ are equations
$x = 3$ is the solution of the equation $2 + x = 5$
$x+9 = 9+x$ is an identity or "true equation"
$(y + 2)^2 = y^2 + 4 + 4y$ is another identity

The part to the left of the equality sign is called the **left member** or **first member**; the part to the right is the **right member** or **second member**.

A **dependent equation** is that with a non-finite set of solutions.

An **inconsistent equation** is that without any solution.

An **identity** (**identical equation** or **true equation**) is an equation which is true for all values of the variables.

Solving equations and rearranging formulae

The **guess-and-check method** (or **trial and error method**) is trying out different values of "x" until you get one that works. Alternatively you might reason a suitable value for "x" **intuitively**, following the clue the equation gives).

The **balance method**: when an equation is not so simple it must be **rearranged using algebra rules** to find an equation which has "x" on its own on one side of an equals sign with a number on the other side.

Properties of the equalities

$a=b \Rightarrow a+c=b+c$
$a=b \Rightarrow a-c=b-c$
$a=b \Rightarrow ac=bc$
$a=b \Rightarrow \frac{a}{c}=\frac{b}{c}$

Properties of adition and product

$a+b=b+a$ conmutative
$a \cdot b=b \cdot a$ conmutative
$a+(b+c)=(a+b)+c$ associative
$a \cdot (b \cdot c)=(a \cdot b) \cdot c$ associative
$a \cdot (b+c)=a \cdot b+a \cdot c$ distributive

Algebra rules

To **move a term from one side of the equals sign to the other**, you have to do the same thing on both sides of the equals sign ("change the side, change the sign").

Multiplication distributes over addition or subtraction to **multiply out brackets.**

Division of equal quantities by the same number produces equal amounts.

You have to first make sure **all of the variables are on the same side** and then you have to **collect like terms.**

These simple steps allow to solve first-degree equations

-Multiply out any brackets

-Get everything off the bottom by changing all terms of both sides to have the same denominator. After that, common denominator can be erased.

-Collect those terms with an "x" in (**subject terms**) on one side of the "=" and all the others on the ther side, reversing the +/- sign if a term crosses the "=".

-Combine like terms on each side of the equation, and reduce it to the form "Ax = B".

-Slide "A" underneath the "B" and divide.

Examples:

$$\frac{3(x+3)}{2}-2(2-3x)=8x-1-2(x+3)$$
$$\frac{3x+9}{2}-4+6x=8x-1-2x-6$$
$$\frac{3x+9}{2}-\frac{8}{2}+\frac{12x}{2}=\frac{16x}{2}-\frac{2}{2}-\frac{4x}{2}-\frac{12}{2}$$
$$3x+9-8+12x=16x-2-4x-12$$
$$3x+12x-16x+4x=-9+8-2-12$$
$$3x=-15$$
$$x=\frac{-15}{3}$$
$$x=-5$$

$$\frac{x+7}{2}-\frac{7-x}{6}=\frac{x-7}{12}+7$$
$$\frac{6x+42}{12}-\frac{14-2x}{12}=\frac{x-7}{12}+\frac{84}{12}$$
$$6x+42-(14-2x)=x-7+84$$
$$6x+42-14+2x=x-7+84$$
$$6x+2x-x=-42+14-7+84$$
$$7x=49$$
$$x=\frac{49}{7}$$
$$x=7$$

With them it is also possible to change the subject of the equation or formula, when two or more variables appear (**rearranging formulae**).

Example: $y=\dfrac{4a(x^2+b)}{3}$

$$y=\frac{4ax^2+4ab}{3}$$

$$\frac{3y}{3}=\frac{4ax^2+4ab}{3}$$

$$3y=4ax^2+4ab$$

$$3y-4ab=4ax^2$$

$$\frac{3y-4ab}{4a}=x$$

finally square root both sides $\sqrt{\dfrac{3y-4ab}{4a}}=x$

Algebra rules are based on properties, but we apply them without being aware of it.

Examples:

$(2x-3)^2+(x-2)^2=3(x+1)+5x(x-1)$

we use squared brackets and distributive property

$4x^2-12x+9+x^2-4x+4=3x+3+5x^2-5x$

we add/substract to both members the same quantities

$4x^2-12x+9+x^2-4x+4-3x-3-5x^2+5x=0$

we use now conmutative property

$4x^2+x^2-5x^2-12x-4x-3x+5x+9+4-3=0$

we use now associative and distributive properties

$0x^2-14x+10=0$

any number multiplied by zero gives zero

$-14x+10=0$

we add to both members the same quantity 14x

$10=14x$

we divide both members by 14

$\dfrac{10}{14}=x$

that is

$\dfrac{5}{7}=x$

$\dfrac{x}{x-1}+\dfrac{2x}{x+1}=3$

we multiply both members by the same quantities

$(x-1)(x+1)\left(\dfrac{x}{x-1}+\dfrac{2x}{x+1}\right)=(x-1)(x+1)3$

we apply distributive property

$\dfrac{(x-1)(x+1)x}{x-1}+\dfrac{(x-1)(x+1)2x}{x+1}=(x-1)(x+1)3$

we cancel a common factor up and down

$(x+1)x+(x-1)2x=(x-1)(x+1)3$

we use quadratics

$(x+1)x+(x-1)2x=(x^2-1)3$

we apply distributive property

$x^2+x+2x^2-2x=3x^2-3$

we add/substract to both members the same quantities

$x^2+x+2x^2-2x-3x^2+3=0$

we use conmutative property

$x^2+2x^2-3x^2+x-2x+3=0$

we use now associative and distributive properties

$0x^2-x+3=0$

any number multiplied by zero gives zero

$-x+3=0$

Quadratic equations

A **quadratic equation** is a second-degree equation, and can be arranged in the form

$$ax^2 + bx + c = 0 \quad a \neq 0$$

(a,b,c are all constants).

We might solve them by completing the square, but it would not be so easy and direct as using the following methods.

The solutions or roots can be always obtained by using the **quadratic formula**: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

The rooted of the above expression, the quantity $b^2 - 4ac$ is called **discriminant**. It is an indication of the solubility:

-When discriminant is positive there are two different solutions
-When discriminant is zero there is a single solution
-When discriminant is negative the equation does not have any (real) solution

Examples: $4x^2 - 4x + 1 = 0$, $x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \cdot 4 \cdot 1}}{2 \cdot 4} = \frac{4 \pm \sqrt{16-16}}{8} = \frac{4 \pm 0}{8} = \frac{4}{8} = \frac{1}{2}$ a root

$x^2 - 6x + 10 = 0$, $x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4 \cdot 1 \cdot 10}}{2 \cdot 1} = \frac{6 \pm \sqrt{36-40}}{2} = \frac{6 \pm \sqrt{-4}}{2}$ no roots

$$1 - x(x-3) = 4x - 1$$
$$1 - x^2 + 3x = 4x - 1$$
$$1 - x^2 + 3x - 4x + 1 = 0$$
$$1 - x^2 + 3x - 4x + 1 = 0$$
$$-x^2 - x + 2 = 0$$

$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \cdot (-1) \cdot 2}}{2 \cdot (-1)} = \frac{1 \pm \sqrt{1-(-8)}}{-2} = \frac{1 \pm \sqrt{9}}{-2} = \frac{1 \pm 3}{-2} = \begin{cases} \frac{1+3}{-2} = -2 \\ \frac{1-3}{-2} = 1 \end{cases}$ two roots

Note: When discriminant is negative the equation does not have any **real solution**. But when **complex numbers** are studied, it can be explained that the equation has two roots that are complex numbers.

Special cases

Some kinds of quadratic equations can be solved without using the quadratic formula:

-Factorization is suitable to solve equations like $ax^2 + bx = 0$, $a \neq 0$ (equations lacking constant term)

-It is possible to rearrange equations like $ax^2 + c = 0$, $a \neq 0$ (equations lacking first degree term)

Examples: $x + 2x^2 = 0$, $x(1+2x) = 0 \begin{cases} x = 0 \\ 1 + 2x = 0, 1 = 2x, x = \frac{1}{2} \end{cases}$

$8x^2 - 18 = 0$, $8x^2 = 18$, $x^2 = \frac{18}{8}$, $x^2 = \frac{9}{4}$, $x = \frac{\pm 3}{2}$

EXERCISES

1) Solve the following quadratic equations without using the quadratic formula:

a) $(3x+1)(3x-1)+\frac{1}{2}(x-2)^2=1-2x$

b) $\frac{x^2+2}{3}-\frac{x^2+1}{4}=1-\frac{x+7}{12}$

c) $\frac{(2x-1)(2x+1)}{3}+\frac{(x-2)^2}{4}=\frac{3x+4}{6}+\frac{x^2}{3}$

2) Solve the following quadratic equations:

a) $(x+1)^2-3x=3$

b) $(2x+1)^2=1+(x-1)(x+1)$

c) $\frac{(x+1)(x-3)}{2}+x=\frac{x}{4}$

d) $x+\frac{3x+1}{2}-\frac{x-2}{3}=x^2-2$

3) Rearrange, simplify, calculate the discriminant and say how many solutions do these equations have:

a) $(5x-3)^2-5x(4x-5)=5x(x-1)$

b) $\frac{1}{2}x^2-2x+\frac{5}{2}=0$

c) $(x+3)^2-2(3x+6)=0$

d) $\frac{x+1}{2}=x-\frac{2x+3}{4}$

4) Explain the reason why a quadratic function graph can cross the x-axis at two points, or else touch it at one point, or have no intersection with it at any point.

5) Prove that the quadratic formula gives solution for a quadratic equation by plugging the formula into the equation and rearranging

Factorising 2nd degree polynomials

If α and β are the solutions of $ax^2+bx+c=0$, then we can factorise ax^2+bx+c in this way:

$$ax^2+bx+c=a(x-\alpha)(x-\beta)$$

Example: the solutions of $4x^2-10x-6=0$ are $-\frac{1}{2}$ and 3 (use the quadratic formula)

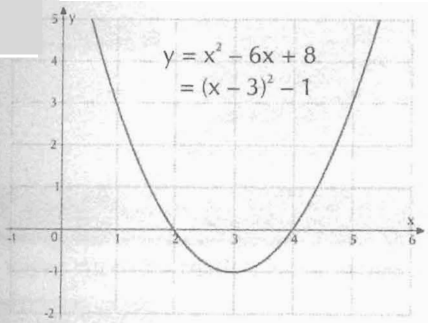
so we can write $4x^2-10x-6=4(x-3)\left(x+\frac{1}{2}\right)$

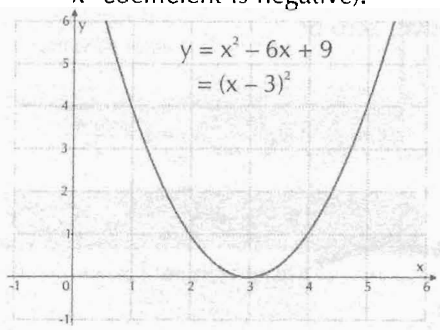
Geometric meaning of the solutions of a quadratic equation

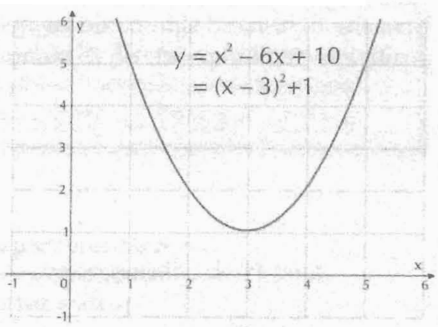
It's good to be able to picture what this means:

A root is just when y = 0, so it's where the graph touches or crosses the x-axis.

$b^2-4ac>0$ Two roots	$b^2-4ac=0$ One root	$b^2-4ac<0$ No roots
So the graph crosses the x-axis twice and these are the roots:	The graph just touches the x-axis from above (or from below if the x^2 coefficient is negative).	The graph doesn't touch the x-axis at all.







Example Find the range of values of k for which: a) f(x)=0 has 2 distinct roots, b) f(x)=0 has 1 root, c) f(x) has no real roots, where $f(x)=3x^2+2x+k$.

First of all, work out what the discriminant is: $b^2-4ac=2^2-4\times3\times k$
$=4-12k$

These calculations are exactly the same. You don't need to do them if you've done a) because the only difference is the equality symbol.

a) Two distinct roots means:
$b^2-4ac>0\Rightarrow 4-12k>0$
$\Rightarrow 4>12k$
$\Rightarrow k<\frac{1}{3}$

b) One root means:
$b^2-4ac=0\Rightarrow 4-12k=0$
$\Rightarrow 4=12k$
$\Rightarrow k=\frac{1}{3}$

c) No roots means:
$b^2-4ac<0\Rightarrow 4-12k<0$
$\Rightarrow 4<12k$
$\Rightarrow k>\frac{1}{3}$

Proving that the quadratic formula gives the solutions of the quadratic equation

By "completing the square"

Say you've got a quadratic equation in standard form... $ax^2+bx+c=0$ ...and you need to find x.

The first thing to do is complete the square.

$$ax^2+bx+c=0$$

$$a\left(x^2+\frac{b}{a}x\right)+c=0$$

$$a\left(x+\frac{b}{2a}\right)^2-\frac{b^2}{4a}+c=0$$

Take a common factor of 'a' out of the first two terms — then it's easier to see how to complete the square.

What you're trying to do is find x — get it on its own on one side.

This is much better, because there's only one x now — so you can rearrange the equation and find what x actually is.

$$a\left(x+\frac{b}{2a}\right)^2=\frac{b^2}{4a}-c=\frac{b^2-4ac}{4a}$$

$$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$$

Take the last two terms over to the right-hand side, and then add them together as fractions.

Divide both sides by a.

Now the right-hand side could be negative, zero or positive — it all depends on a, b and c.

$$x+\frac{b}{2a}=\pm\sqrt{\frac{b^2-4ac}{4a^2}}=\pm\frac{\sqrt{b^2-4ac}}{2a}$$

$$x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

And this is the quadratic formula that you've come to know and love.

Solving equations by means of a substitution

Example 1

Solve $y^2 - 2y - 15 = 0$ and hence solve $(2x + 1)^2 - 2(2x + 1) - 15 = 0$

Solution

$$y^2 - 2y - 15 = 0$$
$$\therefore (y - 5)(y + 3) = 0$$
$$\therefore y - 5 = 0 \text{ or } y + 3 = 0$$
$$\therefore y = 5 \text{ or } y = -3$$

Now, the second equation is exactly the same as the first, except that x has changed its name to $(2x + 1)$. In other words, x has been replaced everywhere by $(2x + 1)$. Accordingly, we change x to $(2x + 1)$ in the answer as well and solve for t.

$$2x + 1 = 5 \text{ or } 2x + 1 = -3$$
$$\therefore 2x = 4 \text{ or } 2x = -4$$
$$\therefore x = 2 \text{ or } x = -2 \text{ Answer}$$

Some **Nasty-looking** equations are just **Quadratics**

$$x^4 + 3x^2 + 6 = 0$$

Arrrgh. How on earth are you supposed to solve something like that? Well the answer is... with great difficulty — that's if you don't spot that you can turn it into quadratic form like this:

$$(x^2)^2 + 3(x^2) + 6 = 0$$

It still looks weird. But, if those x²'s were y's:

$$y^2 + 3y + 6 = 0$$

Now it's a just a simple quadratic that you could solve in your sleep — or the exam, which would probably be more useful.

$$2\sin^2 t - 3\sin t - 2 = 0$$

...it looks hard

$$2(\sin t)^2 - 3(\sin t) - 2 = 0$$

...still looks hard

$$2y^2 - 3y - 2 = 0$$

...looks easy.

Quartic biquadratic equations

A **quartic equation** (**biquadratic equation**) can be arranged to have the expression: $ax^4 + bx^2 + c = 0$.

By substituting $x^2 = t$ our equation turns to a quadratic equation, whose solutions are easily found using the quadratic formula; afterwards, the value of "x" can be got by extracting their roots.

Example: $x^4 - 10x^2 + 9 = 0$

$$x^2 = t$$
$$t^2 - 10t + 9 = 0$$
$$t = \frac{10 \pm \sqrt{100 - 36}}{2} = \begin{cases} 9, x^2 = 9, x = \pm 3 \\ 1, x^2 = 1, x = \pm 1 \end{cases}$$

Example: $2x^6 - 11x^3 + 5 = 0$

1 Spot That It's a Quadratic

Put it in the form: a(something)² + b(same thing) + (number) = 0.

$$2(x^3)^2 - 11(x^3) + 5 = 0$$

Now substitute x^3 for y to make it like a normal quadratic.

2 Substitute

let $x^3 = y \Rightarrow$ $2y^2 - 11y + 5 = 0$

And solve this quadratic to find the values of y.

3 Solve it

$$2y^2 - 11y + 5 = 0$$

$$(2y-1)(y-5) = 0$$

$$y = \frac{1}{2}, \text{ or } 5$$

Now you've got the values of y, you can get the values of x.

4 Find the Original Unknown x

$y = \frac{1}{2}, \text{ or } 5$ but... $y = x^3$ ← This comes from stage 2.

Which means... $x^3 = \frac{1}{2}, \text{ or } 5$

So the answer is... $x = \sqrt{\frac{1}{2}}$ or $\sqrt{\ }$

Disguised Quadratics

1) Put the equation in the
a(something)²+b(same thing)+(number)=0

2) **SUBSTITUTE y for the something in the brackets to get a normal-looking quadratic.**

3) **SOLVE the quadratic in the usual way — i.e. by <u>factorising</u> or using the <u>quadratic formula</u>.**

4) **Stick your answers in the substitution equation to get the values for the ORIGINAL unknown.**

The "zero" rule

$$A \cdot B = 0 \Rightarrow \begin{cases} A = 0 \\ or \\ B = 0 \end{cases}$$

This fact helps us to solve equations in the form of a product of several factors equal to zero.

$$A(x) \cdot B(x) \cdot C(x) \cdot \ldots = 0$$

We must solve each factor's equation

$$A(x) = 0 \qquad B(x) = 0 \qquad C(x) = 0$$

EXERCISES

6) **Solve with a substitution:**

$(3x-2)^4=0$ $(x+5)^2=1$ $(3x-2)^4=16$ $(4-3x)^2=25$

7) Solve:

a) $x^4-5x^2-36=0$ b) $x^4-4x^2+3=0$

c) $25x^4-26x^2+1=0$ d) $x^4-81=0$

e) $x^4-9x^2=0$ f) $9x^4-10x^2+1=0$

8) **Solve applying the "zero" rule:**

a) $(x+3)(x^2-4)=0$ b) $(x-5)(x^2+4)=0$

c) $x(x-1)(2x-3)=0$ d) $3x^2(x+1)^2=0$

9) **Factorise first and solve then:**

a) $x^3-3x^2+2x=0$ b) $x^3+2x^2-x-2=0$

c) $2x^4+6x^3=0$ d) $x^4-6x^3+9x^2=0$

10) Solve:

a) $x^3-27=0$ b) $\frac{64}{x^3}-1=0$

c) $\frac{3x}{5}+\frac{25}{9x^2}=0$ d) $\frac{4}{x}-\frac{x^2}{2}=0$

e) $16x^4-81=0$ f) $\frac{1}{50x}-\frac{25x^3}{2}=0$

Radical equations

A **radical equation** contains one more roots and its rooted has a variable part. In other words, in an radical equation the variable is stuck into a root.

When a term in an equation has a square root (or surd), you should proceed as follows:

Step 1 Isolate the surd on one side of the equation.

Step 2 Square both sides of the equation.

Step 3 Solve the equation.

Step 4 Check your solutions.

It is very important (when you are checking your answers) to realise that the square root of a number is always positive.

For example, $\sqrt{100} = +10$ (not -10)

Example 1

Solve $\sqrt{x+1} - x + 5 = 0$

Solution

$\sqrt{x+1} - x + 5 = 0$

$\therefore \sqrt{x+1} = x - 5$ (Isolating the square root)

$\therefore (\sqrt{x+1})^2 = (x-5)^2$ (Squaring both sides)

$\therefore x + 1 = x^2 - 10x + 25$

$\therefore 0 = x^2 - 11x + 24$

$\therefore 0 = (x-8)(x-3)$

$\therefore x = 8$ or $x = 3$

Check $x = 8 \Rightarrow \sqrt{8+1} - 8 + 5 = 0 \Rightarrow 3 - 8 + 5 = 0$ **Correct!**

$x = 3 \Rightarrow \sqrt{3+1} - 3 + 5 = 0 \Rightarrow 2 - 3 + 5 \neq 0$ **Wrong!**

Answer $x = 8$

Examples:

$\sqrt{x^2+7} + 2 = 2x$

$\sqrt{x^2+7} = 2x - 2$

$\left(\sqrt{x^2+7}\right)^2 = (2x-2)^2$

$x^2 + 7 = 4x^2 + 4 - 8x$

$0 = 3x^2 - 8x - 3$

$x = \frac{8 \pm \sqrt{64+36}}{6} = \frac{8 \pm 10}{6} = \begin{cases} x = 3 \\ x = \frac{-1}{3} \end{cases}$

checking x=3

$\sqrt{3^2+7} + 2 = 2 \cdot x$

$4 + 2 = 2 \cdot 3$ true, so 3 is a solution

checking $x = \frac{-1}{3}$

$\sqrt{\left(\frac{-1}{3}\right)^2 + 7} + 2 = 2 \cdot \frac{-1}{3}$

$\frac{8}{3} + 2 = 2 \cdot \frac{-1}{3}$

$\frac{14}{3} = \frac{-2}{3}$ false, so $\frac{-1}{3}$ is not a solution

Note: $\sqrt{7} + 2x = 5$ is not a radical equation

EXERCISES

11) Solve these equations (and check your answers!):

	8. $\sqrt{3x+1} = x-3$
1. $\sqrt{x+3} = 2$	9. $x+2 = \sqrt{9x}$
2. $\sqrt{x+1} = 4$	10. $\sqrt{5x+4} = x-2$
3. $\sqrt{2x+1} = 5$	11. $\sqrt{4x-3}+2-x = 0$
4. $\sqrt{3x-5} = 2$	12. $\sqrt{12x+1} = 2x-9$
5. $\sqrt{x-1} = x-3$	13. $2\sqrt{7x+1} = x+7$
6. $\sqrt{2x+1} = x-1$	14. $3\sqrt{2x+3} = x+4$
7. $\sqrt{x} = x-6$	15. $\sqrt{x^2+9}-2x+3 = 0$

12) Solve:

a) $x-\sqrt{x} = 2$ b) $x-\sqrt{25-x^2} = 1$

c) $x-\sqrt{169-x^2} = 17$ d) $x+\sqrt{5x+10} = 8$

13) Solve:

a) $(x^2-5x)^2-4(x^2-5x)-12=0$

b) $2x^4-5x^2-7=0$

c) $2(x^2-1)+3(2-x^2)=\frac{-45}{x^2}$

d) $\frac{x^2-1}{35}=\frac{1}{x^2+1}$

e) $\left[(x+1)^2+1\right]^2=4x(x^2+2)+3x^2+10$

f) $x^6-7x^3-8=0$

g) $x^5-10x^3-11x=0$

h) $2\left(\frac{x}{x+1}\right)^2-5\left(\frac{x}{x+1}\right)=0$

14) Solve:

$(x-3)^2=4$ $(2x-1)^2=0$ $(x+2)^3=0$ $(x+2)^3=-8$

$(x+5)^2=1$ $(4-3x)^2=25$ $(3x-2)^4=0$ $(3x-2)^4=16$

15) Solve:

a) $\sqrt{x}+1=\sqrt{3(x-1)}$ b) $\sqrt{x^2-6x+9}-8=0$ c) $\sqrt{x^2-16}=\frac{\sqrt{16x+1}}{3}$

d) $\frac{\sqrt{x}-2}{\sqrt{x}+2}=\frac{\sqrt{x}+1}{\sqrt{x}-3}$ e) $\sqrt{12+\sqrt{x+7}}=\sqrt{25-x}$ f) $2\sqrt{4x+1}=5\sqrt{3x-2}-4$

g) $\left(\sqrt{x-1}-1\right)^2+2=2\sqrt{x-1}$ h) $\sqrt{x-2}+\sqrt{x+1}=\sqrt{x+6}$

i) $x^2+3=\sqrt{2x^4+3x^2+5}$ j) $x+5=\sqrt{x^2+10x+5\sqrt{5(x^2+4)}}$

k) $\sqrt{x-85}-\sqrt[4]{3x-47}=0$ l) $\sqrt{x+3}+\sqrt{x+6}=\frac{3}{\sqrt{x+3}}$

Equations with algebraic fractions

We apply operations and remove denominators provided they are all equal and not zero.

Example: $\frac{1}{x}+3=\frac{x-3}{2x}$

$$\frac{2}{2x}+\frac{6x}{2x}=\frac{x-3}{2x}$$

$$2+6x=x-3$$

$$6x-x=-3-2$$

$$5x=-5$$

$$x=\frac{-5}{5}$$

$$x=-1$$

EXERCISES

16)

1. Solve $x - 7 + \frac{10}{x} = 0$
2. Solve $x - \frac{3}{x} = 2$
3. Solve $x = \frac{5}{x} - \frac{1}{2}$
4. Solve $\frac{3x-5}{x-1} = \frac{2x+15}{x+5}$
5. Solve $\frac{4x+11}{x+9} = \frac{5x+4}{2x+4}$
6. Find the value of x if $(2x-5) + \frac{1}{2x-3} = 0.$
7. Solve for x: $2(x+1) + \frac{6}{x+1} = 13$
8. Find the values of x which satisfy the equation $\frac{2x+3}{x+3} + \frac{x-1}{x+1} = 2$
9. Solve for x: $\frac{3x-1}{3} - \frac{4(x-1)}{5} = \frac{1}{x+1}$
10. Solve $\frac{1}{x+2} + \frac{3}{x-1} = \frac{3x^2+1}{(x+2)(x-1)}$
11. Solve $\frac{4}{x-3} \quad \frac{5}{x+1} = \frac{x^2-1}{(x+1)(x-3)}$
12. Solve for x: $\frac{3}{x-2} - \frac{45}{x^2-4} - \frac{14}{x+2} + 6 = 0$

17) $\begin{cases} x + 3y = 9 \\ \frac{x^2 - 2y + 3}{x-1} = 3 + x \end{cases}$ Rearrange the equations in the form $\begin{cases} ax + by = c \\ a'x + b'y = c' \end{cases}$
Then solve by means of the three methods you have studied last course. Plot both lines and mark the point at which both graphs meet.

18) Add the same number to 100 and to 20 so as to make the greater number three times the smaller one. (Diofanto, 3rd century)

A man who was walking on a road came across a group of people and said to them: "if you were the ones you are plus that number again, plus half the half the half of that number, plus me, we would be 100" How many people were in the group?
(Beda, 8th century)

A man dies, leaving a pregnant wife and his will explain how his estate, 98 gold coins, is to be divided between the wife and a son or a daughter: if the baby is a boy, he will receive two times the mother's part; if the baby is a girl, she will receive half the mother's part. The wife produces twins, one boy and one girl. How is the estate to be divided?
(Chuquet, 15 th century)

Simultaneous equations

A **set of simultaneous equation** consist of two or more equations. They usually have more than one **variables** or **unknowns**.

A set of values for the variables is **a solution of the simultaneous equations** when all the equations become true if the values replace the variables.

Solving simultaneous equations means finding the answers to several equations at the same time.

Verifying solutions

A solution of a set of simultaneous equations can be verified by substituting the values for the variables.

Classifying sets of simultaneous equations

According to the **number of equations and variables** (2, 3, 4, ... equations; 1, 2, 3, 4, ... variables)
According to the **type and degree of the equations** (polynomic, linear, non-linear, rational, irrational...)
According to the **number of solutions**

Number of solutions of two simultaneous linear equations

An **independent system of linear equations** is that with a single solution.

A **dependent system of linear equations** is that with a non-finite set of solutions. One equation is really just another copy of the other: coefficients and constant term are proportional $\frac{a}{a'}=\frac{b}{b'}=\frac{c}{c'}$

An **inconsistent system of linear equations** is that without any solution. That happens when coefficients are proportional but the constant terms are not in the same proportion as coefficients $\frac{a}{a'}=\frac{b}{b'}\neq\frac{c}{c'}$

Linear simultaneous equations with two unknowns

The first type of simultaneous equations we are going to solve is a set of two first-degree equations (**linear equations**) in two variables. That is any set of simultaneous equations equivalent to one like this:

$$\begin{cases} ax+by=c \\ a'x+b'y=c' \end{cases}$$

We must find values for "x" and "y" for which both equations are true.

Solving two linear simultaneous equations with two unknowns

Always before applying any method:

-Rearrange both equations into the above form.

Always after applying any method:

-Once you get the value of one of the unknowns, substitute it into any equation to get the other.

The **substitution method**:

-Leave one variable alone rearranging the easiest equation.
-Plug in the resulting expression for that variable into the other equation.
-Solve the equation obtained.

Another **method**:

-Isolate one variable rearranging the first equation.
-Isolate the same variable rearranging the second equation.
-Form a new equation equalling both expressions and solve it.

The **elimination method**:

-If both coefficients of "x" are opposite numbers (or both coefficients of "y" are) add the equations and solve the resulting equation.
-If not, try to reach that situation multiplying one of the equations by a suitable number.
-Sometimes you have to change both equations multiplying each one by a different number (the coefficient of the like term in the other equation or else its opposite).

Simultaneous equations of second degree

Substitution method is to be used.

Example: $\begin{cases} y = x+1 \\ x^2 + y^2 = 5 \end{cases}$

$$x^2 + (x+1)^2 = 5$$
$$x^2 + x^2 + 2x + 1 = 5$$
$$2x^2 + 2x - 4 = 0$$
$$x^2 + x - 2 = 0$$
$$x = \frac{-1 \pm \sqrt{1^2 - 4\cdot 1 \cdot (-2)}}{2\cdot 1} = \frac{-1 \pm \sqrt{1+8}}{2} = \frac{-1\pm 3}{2} = \begin{cases} x = 1, y = 2 \\ x = -2, y = -1 \end{cases}$$

Use Substitution if one equation is Quadratic

Example:

$-x + 2y = 5$ — (L) ← The linear equation — with only x's and y's in.

$x^2 + y^2 = 25$ — (Q) ← The quadratic equation — with some x^2 and y^2 bits in.

Rearrange the linear equation so that either x or y is on its own on one side of the equals sign.

(L) $-x + 2y = 5$

$\Rightarrow x = 2y - 5$

Substitute this expression into the quadratic equation..

Sub into (Q): $x^2 + y^2 = 25$

$\Rightarrow (2y-5)^2 + y^2 = 25$

...and then rearrange this into the form $ax^2 + bx + c = 0$, so you can solve it — either by factorising or using the quadratic formula.

$\Rightarrow (4y^2 - 20y + 25) + y^2 = 25$

$\Rightarrow 5y^2 - 20y = 0$

$\Rightarrow 5y(y-4) = 0$

$\Rightarrow y = 0$ or $y = 4$

Finally put both these values back into the linear equation to find corresponding values for x:

When y = 0: $-x + 2y = 5$ (L)

$\Rightarrow x = -5$

When y = 4: $-x + 2y = 5$ (L)

$\Rightarrow -x + 8 = 5$

$\Rightarrow x = 3$

So the solutions to the simultaneous equations are: x = –5, y = 0 and x = 3, y = 4.

As usual, check your answers by putting these values back into the original equations.

One Quadratic and One Linear Eqn

1) **Isolate variable in linear equation**
Rearrange the linear equation to get either x or y on its own.

2) **Substitute into quadratic equation**
— to get a quadratic equation in just one variable.

3) **Solve to get values for one variable**
— either by factorising or using the quadratic formula.

4) **Stick these values in the linear equation**
— to find corresponding values for the other variable.

Geometric interpretation of a system of equations' solutions

Two Solutions — Two points of Intersection

Example:

$y = x^2 - 4x + 5$ — ①

$y = 2x - 3$ — ②

Solution:

Substitute expression for y from ② into ①: $2x - 3 = x^2 - 4x + 5$

Rearrange and solve: $x^2 - 6x + 8 = 0$

$(x-2)(x-4) = 0$

$x = 2$ or $x = 4$

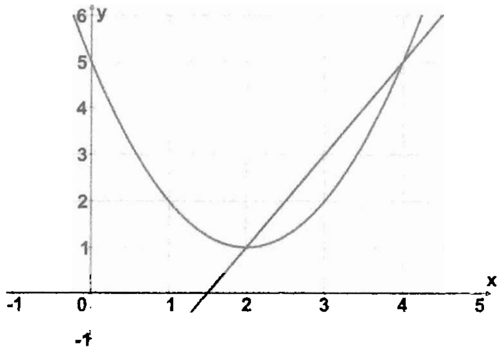
In ② gives: $x = 2 \Rightarrow y = 2\times2 - 3 = 1$

$x = 4 \Rightarrow y = 2\times4 - 3 = 5$

There's 2 pairs of solutions: x=2, y=1 and x=4, y=5

Geometric Interpretation:

So from solving the simultaneous equations, you know that the graphs meet in two places — the points (2,1) an



One Solution — One point of Intersection

Example:

$y = x^2 - 4x + 5$ — ①

$y = 2x - 4$ — ②

Solution:

Substitute ② in ①: $2x - 4 = x^2 - 4x + 5$

Rearrange and solve: $x^2 - 6x + 9 = 0$

$(x-3)^2 = 0$

$x = 3$

Double root i.e. you only get 1 solution from the quadratic.

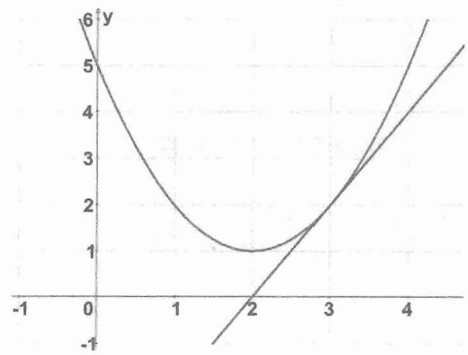
In Equation ② gives: $y = 2\times3 - 4$

$y = 2$

There's 1 solution: x=3, y=2

Geometric Interpretation:

Since the equations have only one solution, the two graphs only meet at one point — (3,2). The straight line is a tangent to the curve.



No Solutions means the Graphs Never Meet

Example:

$y = x^2 - 4x + 5$ — ①

$y = 2x - 5$ — ②

Solution:

Substitute ② in ①: $2x - 5 = x^2 - 4x + 5$

Rearrange and try to solve with the quadratic formula: $x^2 - 6x + 10 = 0$

$b^2 - 4ac = (-6)^2 - 4.10$

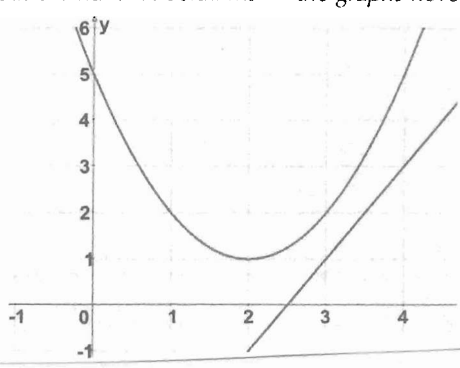
$= 36 - 40 = -4$

$b^2 - 4ac < 0$, so the quadratic has no roots.

So the simultaneous equations have no solutions.

Geometric Interpretation:

The equations have no solutions — the graphs never meet



EXERCISES

19) Solve:

a) $\begin{cases} x+y=1 \\ xy+2y=2 \end{cases}$ b) $\begin{cases} 2x+y=3 \\ x^2+y^2=2 \end{cases}$

c) $\begin{cases} 2x+y=3 \\ xy-y^2=0 \end{cases}$ d) $\begin{cases} x-y=1 \\ x^2+y^2=11-3x \end{cases}$

20) Solve:

a) $\begin{cases} 2x+y=2 \\ xy-y^2=0 \end{cases}$ b) $\begin{cases} \dfrac{x+y}{2}-x=1 \\ \dfrac{x-y}{2}+x^2=0 \end{cases}$

21) Solve:

a) $\begin{cases} x+y=5 \\ x^2y^2=36 \end{cases}$ b) $\begin{cases} y^2-2y+1=x \\ \sqrt{x}+y=5 \end{cases}$

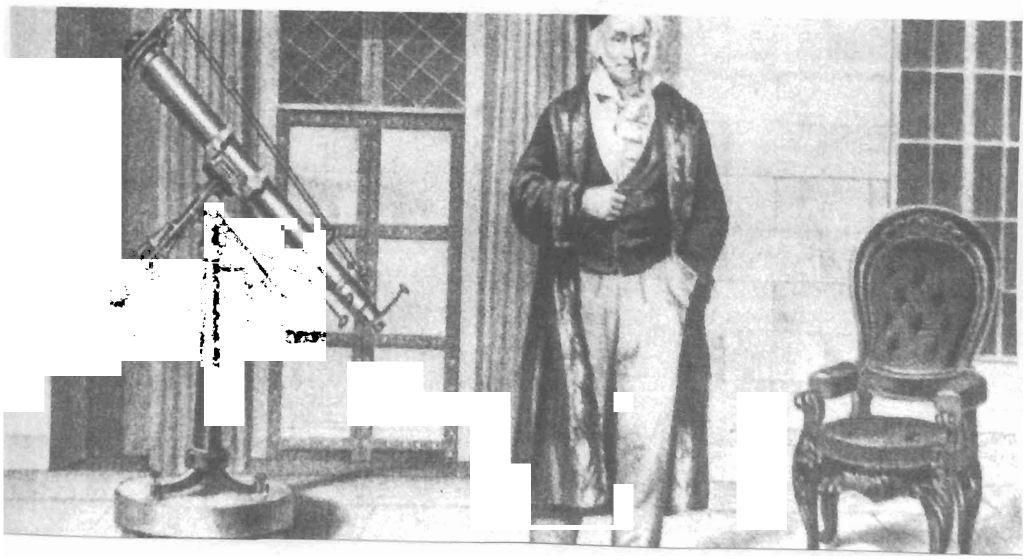
c) $\begin{cases} x^2+y^2=34 \\ x^2-y^2=16 \end{cases}$ d) $\begin{cases} 2\sqrt{x+1}=y+1 \\ 2x-3y=1 \end{cases}$

22) A king sends 28,000 gold coins to the captain for he must pay 7000 infantryman and 7000 knights. With 100 coins he can pay 18 infantrymen more than knights. If there were 1700 infantrymen and 200 knights to be paid how many coins would be needed?
(Cardano, 17th century)

Zaid is said to be given 10 coins less than the square root of Amrou's coins.
Amrou is said to be given 5 coins less than the square root of Zaid's coins.
How many coins will each one receive?
(Beha Eddin, 16th century)

23) Solving for a variable each equation, and joining the expressions two by two with an equal sign, get a pair of simultaneous equations. Solve them afterwards. Finally, find out the value of the variable left

a) $\begin{cases} x+y+z=2 \\ 2x+3y+5z=11 \\ x-5y+6z=29 \end{cases}$ b) $\begin{cases} 2x+y-z=4 \\ -x+2y+3z=3 \\ x+5y+z=14 \end{cases}$ c) $\begin{cases} x-3y-2z=3 \\ 5x-2y+z=-3 \\ 3x+y-z=9 \end{cases}$



Carl Friedrich Gauss (1777–1855) is one of the greatest and most prolific mathematicians of all time. He was born into a poor family in Germany; his father was a gardener and his mother a maid. Gauss learned to calculate before he could talk.

When he was four he corrected his father's calculations of a wage packet.
On his first day at school (at the age of eight) he was set the task of adding $1 + 2 + 3 + \ldots + 98 + 99 + 100$. He came up with the answer 5050 in an instant! (Can you see how he did it? Add $1 + 100$, then $2 + 99$, then $3 + 98$ etc.)

Gauss made numerous discoveries in mathematics and astronomy, including 'Gaussian elimination', which is a method of solving simultaneous equations.

Ex nple:

$$\begin{cases} 2x + y - z = 8 \quad (L_1) \\ -3x - y + 2z = -11 \quad (L_2) \\ -2x + y + 2z = -3 \quad (L_3) \end{cases}$$

The algorithm is as follows: eliminate x from all equations below L_1, and then eliminate y from all equations below L_2. This will put the system into triangular form.
Then, using back-substitution, each unknown can be solved for.

In our example, we eliminate x from L_2 by adding $3/2L_1$ to L_2, and then we eliminate x from L_3 by adding L_1 to L_3.

Formally:

$$L_2 + \frac{3}{2}L_1 \to L_2$$
$$L_3 + L_1 \to L_3$$

The result is:

$$\begin{cases} 2x + y - z = 8 \\ \frac{1}{2}y + \frac{1}{2}z = 1 \\ 2y + z = 5 \end{cases}$$

Now we eliminate y from L_3 by adding $-4L_2$ to L_3.

Formally:

$$L_3 + -4L_2 \to L_3$$

The result is:

$$2x + y - z = 8$$
$$2^{\circ} \cdot \frac{1}{2}z = 1$$
$$-z = 1$$

This result is a system of linear equations in triangular form, and so the first part of the algorithm is complete.

The second part, back-substitution, consists of solving for the unknowns in reverse order.

EXERCISES

24) Two cyclists start at the same time from opposite ends of a course that is 60 miles long. One cyclist is riding at a speed that is 4/5 of the second cyclist's one. How long after will they meet? What are their respective speeds?

25) A person buys a radio and a computer with 2,500 €. Then he/she sells them and gets 2,157.50€
The loss in the radio was 10% of its original value.
The loss in the computer was 15% of its original value.
How much did each object cost at the beginning?

26) The qualification for a job is obtained through two tests: a written test (worth the 65%) and a spoken test (worth the 35%). Someone gets 12 marks adding up the marks of the two exams and his/her qualification is 5.7. How many marks has he/she got in each test?

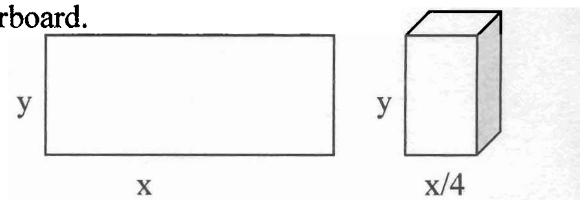
27) The sides of a triangle are 18, 16 and 9 cm long. If you substract the same quantity from each side you will get a right-angled triangle. What is this quantity?

28) One pipe can fill a pool two times faster than a second pipe. When both pipes are opened, they fill the pool in three hours. How long would it take to fill the pool if only the slower pipe is used? And if only the faster pipe is used?

29) Calculate the measures of a rectangle:
The diagonal is 75 m long; the rectangle is similar to one with length 48 m and width 36 m.

30) If you shorten the basis of a rectangle in 2 cm and you shorten the height of it in 1 cm, its area will decrease in 13 cm^2. The rectangle perimeter is 24 cm.
Calculate the measures of the rectangle.

31) We build a box without basis with a piece of cardboard of 240 cm^2. If the volume of the box is 360 cm^2, find out the measures of the piece of carboard.



32) How many litres of a 70% alcohol solution must be added to 50 litres of a 40% alcohol solution to produce a 50% alcohol solution?

33) This time, suppose you work in a lab. You need a 15% acid solution for a certain test, but your supplier ships a 10% solution and a 30% solution. Rather than pay the hefty surcharge to have the supplier make a 15% solution, you decide to mix 10% solution with 30% solution to make your own 15% solution. You need 10 litres of the 15% acid solution. How many litres of 10% solution and 30% solution should you use?

34) You have $50,000 to invest, and two funds that you'd like to invest in.
The You-Risk-It Fund yields 14% interest. The Extra-Dull Fund yields 6% interest.
Because of taxation implications, you don't think you can afford to earn more than $4,500 in interest income this year. How much should you put in each fund?

Inequalities

Here is a true mathematical statement: $5 > 2$

Add 1 to both sides and it is still true: $6 > 3$

Divide both sides by 3 and it is still true: $2 > 1$

Multiply both sides by 10 and it is still true: $20 > 10$

Subtract 5 from both sides and it is still true: $15 > 5$

Multiply both sides by -1 and it is **false!:** $-15 > -5$

But if we reverse the inequality sign, it is true again!: $-15 < -5$

So we can generalize:

$a < b \Rightarrow a + c < b + c$

$a < b \Rightarrow a - c < b - c$

$\left.\begin{matrix} a < b \\ c > 0 \end{matrix}\right\} \Rightarrow ac < bc$

$\left.\begin{matrix} a < b \\ c > 0 \end{matrix}\right\} \Rightarrow \frac{a}{c} < \frac{b}{c}$

$\left.\begin{matrix} a < b \\ c < 0 \end{matrix}\right\} \Rightarrow ac > bc$

$\left.\begin{matrix} a < b \\ c < 0 \end{matrix}\right\} \Rightarrow \frac{a}{c} > \frac{b}{c}$

$a > b \Rightarrow a + c > b + c$

$a > b \Rightarrow a - c > b - c$

$\left.\begin{matrix} a > b \\ c > 0 \end{matrix}\right\} \Rightarrow ac > bc$

$\left.\begin{matrix} a > b \\ c > 0 \end{matrix}\right\} \Rightarrow \frac{a}{c} > \frac{b}{c}$

$\left.\begin{matrix} a > b \\ c < 0 \end{matrix}\right\} \Rightarrow ac < bc$

$\left.\begin{matrix} a > b \\ c < 0 \end{matrix}\right\} \Rightarrow \frac{a}{c} < \frac{b}{c}$

$a \le b \Rightarrow a + c \le b + c$

$a \le b \Rightarrow a - c \le b - c$

$\left.\begin{matrix} a \le b \\ c > 0 \end{matrix}\right\} \Rightarrow ac \le bc$

$\left.\begin{matrix} a \le b \\ c > 0 \end{matrix}\right\} \Rightarrow \frac{a}{c} \le \frac{b}{c}$

$\left.\begin{matrix} a \le b \\ c < 0 \end{matrix}\right\} \Rightarrow ac \ge bc$

$\left.\begin{matrix} a \le b \\ c < 0 \end{matrix}\right\} \Rightarrow \frac{a}{c} \ge \frac{b}{c}$

$a \ge b \Rightarrow a + c \ge b + c$

$a \ge b \Rightarrow a - c \ge b - c$

$\left.\begin{matrix} a \ge b \\ c > 0 \end{matrix}\right\} \Rightarrow ac \ge bc$

$\left.\begin{matrix} a \ge b \\ c > 0 \end{matrix}\right\} \Rightarrow \frac{a}{c} \ge \frac{b}{c}$

$\left.\begin{matrix} a \ge b \\ c < 0 \end{matrix}\right\} \Rightarrow ac \le bc$

$\left.\begin{matrix} a \ge b \\ c < 0 \end{matrix}\right\} \Rightarrow \frac{a}{c} \le \frac{b}{c}$

Inequalities are very like equations.

We may add the same number to both sides: $x > y \Rightarrow x + 3 > y + 3$

We may subtract the same number from both sides: $x > y \Rightarrow x - 2 > y - 2$

We may multiply both sides by a positive number: $x < y \Rightarrow 5x < 5y$

We may divide both sides by a positive number: $x < y \Rightarrow \frac{x}{2} < \frac{y}{2}$

But! If you multiply (or divide) both sides by a negative number, the inequality sign changes around:

$x < y \Rightarrow -x > -y$ and $a \ge b \Rightarrow -a \le -b$

Linear inequalities can be solved by rearrangement in much the same way as linear equations. However, care must be taken to reverse the direction of the inequality when multiplying or dividing by a negative.

Example Solve the inequality $8 - 3x > 23$.

Subtract 8 from both sides: $-3x > 15$

Divide both sides by -3: $x < -5$.

Example Solve the inequality $5x - 3 > 3x - 10$.

Subtract $3x$ from both sides: $2x - 3 > -10$

Add 3 to both sides: $2x > -7$

Divide both sides by 2: $x > -3.5$.

Example

Show on the number line the solution set of $1 - 4x < 9$, $x \in R$

Solution

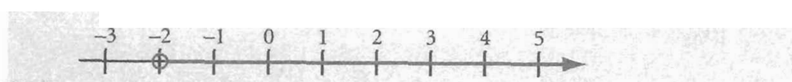
$1 - 4x < 9$

$\therefore -4x < 8$ (Subtracting 1 from both sides)

$\therefore 4x > -8$ (Multiplying both sides by -1 and changing the inequality sign around)

$\therefore x > -2$ (Dividing both sides by 4)

The solution set is all real numbers greater than -2.



Example

Show on the number line the solution set of $-9 \leq 5 - 4x \leq 21$, $x \in Z$

Solution

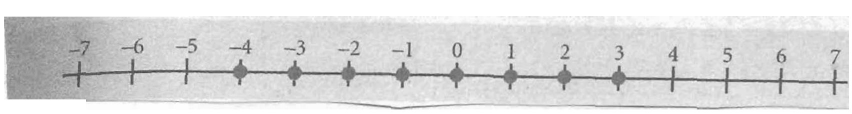
$-9 \leq 5 - 4x \leq 21$

$\therefore -14 \leq -4x \leq 16$ (Subtracting 5 from all three parts)

$\therefore 14 \geq 4x \geq -16$ (Multiplying by -1 and changing the inequality signs around)

$\therefore 3.5 \geq x \geq -4$ (Dividing by 4)

The solution set is all integers between -4 and $3\frac{1}{2}$ (including -4).



EXERCISES

35) Solve these inequalities and show the solution set on the number line in each case:

(i) $2x+1<10,\ x\in N$
(ii) $3x-5>10,\ x\in R$
(iii) $4x+1\le 13,\ x\in R$
(iv) $3x-1\le x+3,\ x\in N$
(v) $1-x>2x-11,\ x\in R$
(vi) $x+3\le 5x-17,\ x\in R$
(vii) $1+x\ge 3x-1,\ x\in N$
(viii) $1-5x>-29,\ x\in R$
(ix) $4\le 1-x,\ x\in R$
(x) $x+8>4(x-1),\ x\in N$

36) Solve these inequalities and show the solutions on the number line in each case:

(i) $\frac{3x-1}{2}<10,\ x\in N$
(ii) $\frac{8x+1}{5}\ge 5,\ x\in R$
(iii) $\frac{11+2x}{7}<3,\ x\in R$
(iv) $\frac{4(x+3)}{3}\ge 12,\ x\in R$
(v) $\frac{x+5}{3}\le\frac{x+11}{5},\ x\in N$
(vi) $\frac{2x-1}{5}\ge\frac{6x-1}{16},\ x\in R$
(vii) $\frac{1}{2}x-1>1,\ x\in R$
(viii) $\frac{1}{2}(x-1)>2,\ x\in R$
(ix) $\frac{1-x}{2}\le\frac{x-1}{3},\ x\in R$
(x) $\frac{3(x-4)+1}{2}<3,\ x\in N$

37) Solve these sets of simultaneous inequalities and say what are the solutions in each case

a) $\begin{cases} x-2>0 \\ x+3>0 \end{cases}$
b) $\begin{cases} 3-x>0 \\ 3+x>0 \end{cases}$
c) $\begin{cases} x+1>0 \\ x-5\le 0 \end{cases}$
d) $\begin{cases} x\ge 0 \\ 1-x<0 \end{cases}$

38) Solve these inequalities and show the solutions on the number line in each case

(i) $5\le 2x+1\le 11,\ x\in N$
(ii) $-3\le 2x-1\le 13,\ x\in Z$
(iii) $-1<3x+2<17,\ x\in R$
(iv) $-9\le 4x-1\le 11,\ x\in Z$
(v) $1\le 2x-7\le 5,\ x\in N$

39) List the elements in these sets:

(i) $\{x \mid 1 - 3x \geq -8,\ x \in N\}$

(ii) $\{x \mid -11 \leq 3 + 7x \leq 31,\ x \in Z\}$

iii) $\{x \mid -14 < 5x + 1 < 26,\ x \in Z\}$

iv) $\{x \mid -8 \leq 7x - 1 \leq 20,\ x \in Z\}$

(v) $\{x \mid -5 \leq 1 - 2x \leq 3,\ x \in Z\}$

40) Show the solutions to the following inequalities on the number line:

(i) $1 \leq \frac{1}{2}x + 1 \leq 2\frac{1}{2},\ x \in N$

(ii) $-3 \leq \frac{1}{4}x + \frac{1}{2} \leq 5,\ x \in R$

(iii) $-1 \leq 1 - \frac{1}{2}x \leq 3,\ x \in Z$

(iv) $-3 \leq \frac{2x - 1}{5} \leq 3,\ x \in R$

(v) $11 < \frac{1 - 7x}{2} < 18,\ x \in R$

Non-linear inequalities

Quadratic inequalities are solved in a similar way to quadratic equations. At the final stage, we can study the signs of the factors and decide which real numbers fulfill the condition of making the quadratic positive or negative.

Inequalities with rational expressions are to be solved in the same way after factorising both numerator and denominator.

Examples: $x^2 - 3x + 2 < 0$

the roots of $x^2 - 3x + 2$ are 1 and 3 (use the quadratic formula)

so we can write $(x-1)(x-2) < 0$

we must find those values of "x" that make the product negative

	when x<1	when x=1	when 1<x<2	when x=2	when 2<x
$(x-1)$	NEGATIVE	ZERO	POSITIVE	POSITIVE	POSITIVE
$(x-2)$	NEGATIVE	NEGATIVE	NEGATIVE	ZERO	POSITIVE
$(x-1)(x-2)$	POSITIVE	ZERO	NEGATIVE	ZERO	POSITIVE

so those values are: numbers smaller than 1 and numbers greater than 2 as well

we can say it more concisely (-∞,1)U(2,∞)

$$\frac{x+3}{2-x} \geq 0$$

we must find those values of "x" that make the quotient positive or zero

	when x < – 3	when x = – 3	when – 3<x<2	when x=2	when 2<x
$(x+3)$	NEGATIVE	ZERO	POSITIVE	POSITIVE	POSITIVE
$(2-x)$	POSITIVE	POSITIVE	POSITIVE	ZERO	NEGATIVE
$\frac{x+3}{2-x}$	NEGATIVE	ZERO	POSITIVE	does not exist	NEGATIVE

so those values are: number – 3 and numbers between – 3 and 2 as well

we can say it more concisely [– 3 , 2)

In quadratic inequalities, a sketch graph is often helpful at the final stage.

Example Solve the inequality $x^2 + 2x - 5 > 0$.

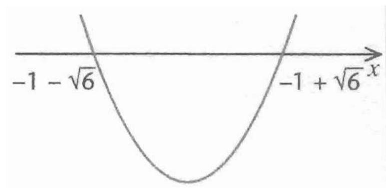
$\sqrt{24} = \sqrt{4 \times 6} = 2\sqrt{6}$.

Use the formula to solve $x^2 + 2x - 5 = 0$. $x = \frac{-2 \pm \sqrt{24}}{2} = \frac{-2 \pm 2\sqrt{6}}{2}$

Simplify the result. $x = -1 \pm \sqrt{6}$

$x^2 + 2x - 5 > 0$ when the curve is above the x-axis.

Sketch the graph of $y = x^2 + 2x - 5$:



Write the solution as two separate inequalities.

From the sketch, $x^2 + 2x - 5 > 0$ when $x < -1 - \sqrt{6}$ or when $x > -1 + \sqrt{6}$.

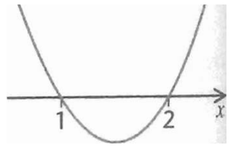
Example Solve the inequality $x^2 - 3x + 2 < 0$.

$y = 0$ when $x = 1$ or when $x = 2$. The graph must cross the x-axis at these points.

$(x - 1)(x - 2) < 0$ when the curve is below the x-axis.

Factorise the quadratic expression: $(x - 1)(x - 2) < 0$.

Sketch the graph of $y = (x - 1)(x - 2)$:



The graph shows that $(x - 1)(x - 2) < 0$ for x values between 1 and 2.

It follows that $x^2 - 3x + 2 < 0$ when $1 < x < 2$.

Example Find the ranges of x which satisfy these inequalities:

① $-x^2 + 2x + 4 \geq 1$

First rewrite the inequality with zero on one side.

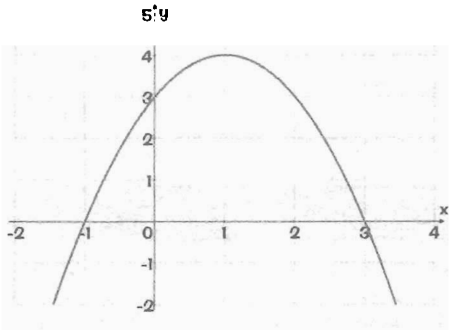
$$-x^2 + 2x + 3 \geq 0$$

Then draw the graph of $y = -x^2 + 2x + 3$:

So find where it crosses the x-axis (i.e. where y=0):

$$-x^2 + 2x + 3 = 0 \Rightarrow x^2 - 2x - 3 = 0$$
$$\Rightarrow (x+1)(x-3) = 0$$
$$\Rightarrow x = -1 \text{ or } x = 3$$

And the coefficient of x^2 is negative, so the graph is n-shaped. So it looks like this:



You're interested in when this is positive or zero, i.e. when it's above the x-axis.

From the graph, this is when x is between −1 and 3 (including those points). So your answer is...

$$-x^2 + 2x + 4 \geq 1 \text{ when } -1 \leq x \leq 3.$$

② $2x^2 - x - 3 > 0$

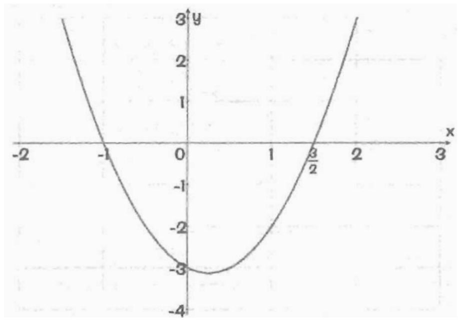
This one already has zero on one side, so draw the graph of $y = 2x^2 - x - 3$.

Find where it crosses the x-axis:

$$2x^2 - x - 3 = 0$$
$$\Rightarrow (2x - 3)(x + 1)$$
$$\Rightarrow x = \tfrac{3}{2} \text{ or } x = -1$$

Factorise it to find the roots.

And the coefficient of x^2 is positive, so the graph is u-shaped. And looks like this:



You need to say when this is positive. Looking at the graph, there are two parts of the x-axis where this is true — when x is less than −1 and when x is greater than 3/2. So your answer is:

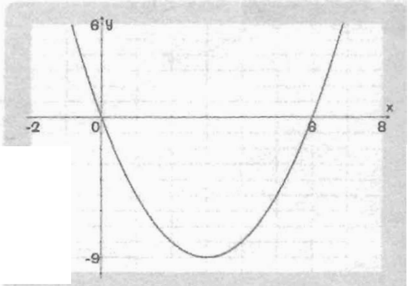
$$2x^2 - x - 3 > 0 \text{ when } x < -1 \text{ or } x > \tfrac{3}{2}.$$

Example Solve $36x < 6x^2$.

$$36x < 6x^2$$
equation 1 ⟹ $\Rightarrow 6x < x^2$
$$\Rightarrow 0 < x^2 - 6x$$

So draw the graph of

$$y = x^2 - 6x = x(x - 6)$$



And this is positive when $x < 0$ or $x > 6$.

If you divide by x in equation 1, you'd only get half the solution — you'd miss the $x < 0$ part.

EXERCISES

41) Study the sign of each factor and find the range of x that satisfies:

a) $(x-1)(x+3)>0$ b) $x(x-4)<0$ e) $(x+3)(x-2)<2$

c) $(x-5)(x+2)\leq 0$ d) $(x+1)(3-x)\leq 0$

42) Solve:

a) $x^2-2x+3>x+1$

b) $-x^2+3x-6<-x-2$

c) $-x^2+x-5\geq -2x-3$

d) $2x<35-x^2$

e) $x^2+6x+9>0$

f) $2x^2-3x-5\leq 0$

43) Solve:

a) $\frac{x^2-9}{5}-\frac{(x+2)(x-2)}{15}<\frac{1-2x}{3}$

b) $\frac{x-1}{2}-\frac{1}{3}>x+\frac{3x-x^2}{3}$

44) Find the values of k, such that

$(x-5)(x-3)>k$ for all possible values of x.

45) Study the sign of each factor and solve:

a) $\frac{2-x}{x-7}\geq 0$ b) $\frac{x^2}{3-x}\geq 0$ c) $\frac{x+1}{x^2}<0$

46) Either algebraically, or by sketching the graphs, solve the inequality

$4x+7>7x+4$

Simultaneous inequalities on two variables

Formulating a linear programming problem

To formulate a linear programming problem you need to:

- Identify the **variables** in the problem and give each one a label.
- Express the **constraints** of the problem in terms of the variables. You need to include non-negativity constraints such as $x \geqslant 0$, $y \geqslant 0$.
- Express the quantity to be optimised in terms of the variables. The expression produced is called the **objective function**.

x and y are often used for the variables.

Typically, this may be to maximise a profit or minimise a loss.

Example

A small company produces two types of armchair. The cost of labour and materials for the two types is shown in the table.

	Labour	Materials
Standard	£30	£25
Deluxe	£40	£50

The total spent on labour must not be more than £1150 and the total spent on materials must not be more than £1250. The profit on a standard chair is £70 and the profit on a deluxe chair is £100.
How many chairs of each type should be made to maximise the profit?

In this case, the variables are the number of chairs of each type that may be produced. Using x to represent the number of standard chairs and y to represent the number of deluxe chairs, the constraints may be written as:

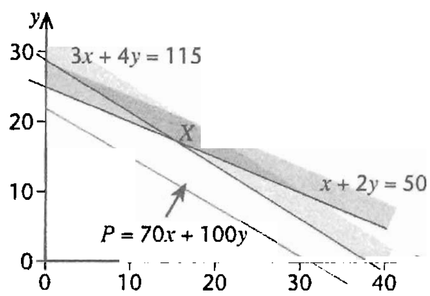
$$30x + 40y \leqslant 1150 \Rightarrow 3x + 4y \leqslant 115$$
$$25x + 50y \leqslant 1250 \Rightarrow x + 2y \leqslant 50$$
and $x \geqslant 0, \quad y \geqslant 0.$

It's a good idea to simplify the constraints where possible.

Using P to stand for the profit, the problem is to maximise $P = 70x + 100y$.

The graphical method of solution

Each constraint is represented by a region on the graph. It's a good idea to shade out the *unwanted* region for each one. The part that remains unshaded then defines the **feasible region** containing the points that satisfy all of the constraints.



The blue line represents the points where the profit takes a particular value. Moving the line in the direction of the arrow corresponds to increasing the profit. This suggests that the maximum profit occurs at the point X.

Solving $3x + 4y = 115$ and $x + 2y = 50$ simultaneously gives X as (15, 17.5).

X does not represent the solution in this case because both x and y must be integers.

The nearest points with integer coordinates in the feasible region are (15,17) and (14,18) The profit, given by $P = 70x + 100y$, is greater at (14,18).

The maximum profit is made by producing 14 standard and 18 deluxe chairs.