

5

FUNCIÓNS E FÓRMULAS TRIGONOMÉTRICAS

Páxina 128

1. Aínda que o método para resolver as seguintes preguntas se sistematiza na páxina seguinte, podes resolvelas agora:

- a) Cantos radiáns corresponden aos 360° dunha circunferencia?
 b) Cantos graos mide 1 radián?
 c) Cantos graos mide un ángulo que ten $\frac{\pi}{2}$ radiáns?
 d) Cantos radiáns equivalen a 270° ?

a) 2π

b) $\frac{360^\circ}{2\pi} = 57^\circ 17' 44,8''$

c) $\frac{360^\circ}{2\pi} \cdot \frac{\pi}{2} = 90^\circ$

d) $\frac{270^\circ}{360^\circ} \cdot 2\pi = 3 \frac{\pi}{2}$

Páxina 129

2. Pasa a radiáns os seguintes ángulos:

a) 30°

b) 72°

c) 90°

d) 127°

e) 200°

f) 300°

Exprésalo o resultado en función de π e despois en forma decimal.

Exemplo: $30^\circ = 30 \cdot \frac{\pi}{180} \text{ rad} = \frac{\pi}{6} \text{ rad} \approx 0,52 \text{ rad}$

a) $\frac{2\pi}{360^\circ} \cdot 30^\circ = \frac{\pi}{6} \text{ rad} \approx 0,52 \text{ rad}$

b) $\frac{2\pi}{360^\circ} \cdot 72^\circ = \frac{2\pi}{5} \text{ rad} \approx 1,26 \text{ rad}$

c) $\frac{2\pi}{360^\circ} \cdot 90^\circ = \frac{\pi}{2} \text{ rad} \approx 1,57 \text{ rad}$

d) $\frac{2\pi}{360^\circ} \cdot 127^\circ \approx 2,22 \text{ rad}$

e) $\frac{2\pi}{360^\circ} \cdot 200^\circ = \frac{10\pi}{9} \text{ rad} \approx 3,49 \text{ rad}$

f) $\frac{2\pi}{360^\circ} \cdot 300^\circ = \frac{5\pi}{3} \text{ rad} \approx 5,24 \text{ rad}$

3. Pasa a graos os seguintes ángulos:

a) 2 rad

b) 0,83 rad

c) $\frac{\pi}{5}$ rad

d) $\frac{5\pi}{6}$ rad

e) 3,5 rad

f) π rad

a) $\frac{360^\circ}{2\pi} \cdot 2 = 114^\circ 35' 29,6''$

b) $\frac{360^\circ}{2\pi} \cdot 0,83 = 47^\circ 33' 19,8''$

c) $\frac{360^\circ}{2\pi} \cdot \frac{\pi}{5} = 36^\circ$

d) $\frac{360^\circ}{2\pi} \cdot \frac{5\pi}{6} = 150^\circ$

e) $\frac{360^\circ}{2\pi} \cdot 3,5 = 200^\circ 32' 6,8''$

f) $\frac{360^\circ}{2\pi} \cdot \pi = 180^\circ$

4. Completa a seguinte táboa e engade as razóns trigonométricas (seno, coseno e tanxente) de cada un dos ángulos. Hache ser útil para a próxima parte:

GRAOS	0°	30°		60°	90°		135°	150°	
RADIÁNS			$\frac{\pi}{4}$			$\frac{2}{3}\pi$			π

GRAOS	210°	225°		270°			330°	360°
RADIÁNS			$\frac{4}{3}\pi$		$\frac{5}{3}\pi$	$\frac{7}{4}\pi$		

La tabla completa está en el siguiente apartado (página siguiente) del libro de texto. Tan solo falta la última columna, que es igual que la primera.

Página 133

1. Demostra a fórmula II.2 a partir da fórmula:

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

$$\cos(\alpha - \beta) = \cos(\alpha + (-\beta)) = \cos \alpha \cos(-\beta) - \sin \alpha \sin(-\beta) =$$

$$= \cos \alpha \cos \beta - \sin \alpha (-\sin \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

2. Demostra a fórmula II.3 a partir da fórmula:

$$\operatorname{tg}(\alpha + \beta) = \frac{\operatorname{tg} \alpha + \operatorname{tg} \beta}{1 - \operatorname{tg} \alpha \operatorname{tg} \beta}$$

$$\operatorname{tg}(\alpha - \beta) = \operatorname{tg}(\alpha + (-\beta)) = \frac{\operatorname{tg} \alpha + \operatorname{tg}(-\beta)}{1 - \operatorname{tg} \alpha \operatorname{tg}(-\beta)} \stackrel{(*)}{=} \frac{\operatorname{tg} \alpha + (-\operatorname{tg} \beta)}{1 - \operatorname{tg} \alpha (-\operatorname{tg} \beta)} = \frac{\operatorname{tg} \alpha - \operatorname{tg} \beta}{1 + \operatorname{tg} \alpha \operatorname{tg} \beta}$$

$$(*) \text{ Como } \left. \begin{array}{l} \sin(-\alpha) = -\sin \alpha \\ \cos(-\alpha) = \cos \alpha \end{array} \right\} \rightarrow \operatorname{tg}(-\alpha) = -\operatorname{tg} \alpha$$

3. Demuestra a fórmula II.3 a partir das seguintes fórmulas:

$$\text{sen}(\alpha - \beta) = \text{sen } \alpha \cos \beta - \cos \alpha \text{sen } \beta$$

$$\text{cos}(\alpha - \beta) = \text{cos } \alpha \cos \beta + \text{sen } \alpha \text{sen } \beta$$

$$\begin{aligned} \text{tg}(\alpha - \beta) &= \frac{\text{sen}(\alpha - \beta)}{\text{cos}(\alpha - \beta)} = \frac{\text{sen } \alpha \cos \beta - \cos \alpha \text{sen } \beta}{\text{cos } \alpha \cos \beta + \text{sen } \alpha \text{sen } \beta} \quad (*) \\ &= \frac{\frac{\text{sen } \alpha \cos \beta}{\text{cos } \alpha \cos \beta} - \frac{\cos \alpha \text{sen } \beta}{\text{cos } \alpha \cos \beta}}{\frac{\text{cos } \alpha \cos \beta}{\text{cos } \alpha \cos \beta} + \frac{\text{sen } \alpha \text{sen } \beta}{\text{cos } \alpha \cos \beta}} = \frac{\text{tg } \alpha - \text{tg } \beta}{1 + \text{tg } \alpha \text{tg } \beta} \end{aligned}$$

(*) Dividimos numerador y denominador por $\text{cos } \alpha \cos \beta$.

4. Se $\text{sen } 12^\circ = 0,2$ e $\text{sen } 37^\circ = 0,6$, determina $\text{cos } 12^\circ$, $\text{tg } 12^\circ$, $\text{cos } 37^\circ$ e $\text{tg } 37^\circ$. Calcula, despois, a partir delas, as razóns trigonométricas de 49° e de 25° , utilizando as fórmulas (I) e (II).

• $\text{sen } 12^\circ = 0,2$

$$\text{cos } 12^\circ = \sqrt{1 - \text{sen}^2 12^\circ} = \sqrt{1 - 0,04} = 0,98$$

$$\text{tg } 12^\circ = \frac{0,2}{0,98} = 0,2$$

• $\text{sen } 37^\circ = 0,6$

$$\text{cos } 37^\circ = \sqrt{1 - \text{sen}^2 37^\circ} = \sqrt{1 - 0,36} = 0,8$$

$$\text{tg } 37^\circ = \frac{0,6}{0,8} = 0,75$$

• $49^\circ = 12^\circ + 37^\circ$, luego:

$$\begin{aligned} \text{sen } 49^\circ &= \text{sen}(12^\circ + 37^\circ) = \text{sen } 12^\circ \cos 37^\circ + \cos 12^\circ \text{sen } 37^\circ = \\ &= 0,2 \cdot 0,8 + 0,98 \cdot 0,6 = 0,748 \end{aligned}$$

$$\begin{aligned} \text{cos } 49^\circ &= \text{cos}(12^\circ + 37^\circ) = \text{cos } 12^\circ \cos 37^\circ - \text{sen } 12^\circ \text{sen } 37^\circ = \\ &= 0,98 \cdot 0,8 - 0,2 \cdot 0,6 = 0,664 \end{aligned}$$

$$\text{tg } 49^\circ = \text{tg}(12^\circ + 37^\circ) = \frac{\text{tg } 12^\circ + \text{tg } 37^\circ}{1 - \text{tg } 12^\circ \text{tg } 37^\circ} = \frac{0,2 + 0,75}{1 - 0,2 \cdot 0,75} = 1,12$$

(Podría calcularse $\text{tg } 49^\circ = \frac{\text{sen } 49^\circ}{\text{cos } 49^\circ}$).

• $25^\circ = 37^\circ - 12^\circ$, luego:

$$\begin{aligned} \text{sen } 25^\circ &= \text{sen}(37^\circ - 12^\circ) = \text{sen } 37^\circ \cos 12^\circ - \cos 37^\circ \text{sen } 12^\circ = \\ &= 0,6 \cdot 0,98 - 0,8 \cdot 0,2 = 0,428 \end{aligned}$$

$$\begin{aligned} \text{cos } 25^\circ &= \text{cos}(37^\circ - 12^\circ) = \text{cos } 37^\circ \cos 12^\circ + \text{sen } 37^\circ \text{sen } 12^\circ = \\ &= 0,8 \cdot 0,98 + 0,6 \cdot 0,2 = 0,904 \end{aligned}$$

$$\text{tg } 25^\circ = \text{tg}(37^\circ - 12^\circ) = \frac{\text{tg } 37^\circ - \text{tg } 12^\circ}{1 + \text{tg } 37^\circ \text{tg } 12^\circ} = \frac{0,75 - 0,2}{1 + 0,75 \cdot 0,2} = 0,478$$

5. Demostra a seguinte igualdade:

$$\frac{\cos(a+b) + \cos(a-b)}{\operatorname{sen}(a+b) + \operatorname{sen}(a-b)} = \frac{1}{\operatorname{tg} a}$$

$$\begin{aligned} \frac{\cos(a+b) + \cos(a-b)}{\operatorname{sen}(a+b) + \operatorname{sen}(a-b)} &= \frac{\cos a \cos b - \operatorname{sen} a \operatorname{sen} b + \cos a \cos b + \operatorname{sen} a \operatorname{sen} b}{\operatorname{sen} a \cos b + \cos a \operatorname{sen} b + \operatorname{sen} a \cos b - \cos a \operatorname{sen} b} = \\ &= \frac{2 \cos a \cos b}{2 \operatorname{sen} a \cos b} = \frac{\cos a}{\operatorname{sen} a} = \frac{1}{\operatorname{tg} a} \end{aligned}$$

6. Demostra as tres fórmulas (III.1), (III.2) e (III.3) facendo $\alpha = \beta$ nas fórmulas (I).

$$\operatorname{sen} 2\alpha = \operatorname{sen}(\alpha + \alpha) = \operatorname{sen} \alpha \cos \alpha + \cos \alpha \operatorname{sen} \alpha = 2 \operatorname{sen} \alpha \cos \alpha$$

$$\cos 2\alpha = \cos(\alpha + \alpha) = \cos \alpha \cos \alpha - \operatorname{sen} \alpha \operatorname{sen} \alpha = \cos^2 \alpha - \operatorname{sen}^2 \alpha$$

$$\operatorname{tg} 2\alpha = \operatorname{tg}(\alpha + \alpha) = \frac{\operatorname{tg} \alpha + \operatorname{tg} \alpha}{1 - \operatorname{tg} \alpha \operatorname{tg} \alpha} = \frac{2 \operatorname{tg} \alpha}{1 - \operatorname{tg}^2 \alpha}$$

7. Determina as razóns trigonométricas de 60° a partir das de 30° .

$$\operatorname{sen} 60^\circ = \operatorname{sen}(2 \cdot 30^\circ) = 2 \operatorname{sen} 30^\circ \cos 30^\circ = 2 \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$$

$$\cos 60^\circ = \cos(2 \cdot 30^\circ) = \cos^2 30^\circ - \operatorname{sen}^2 30^\circ = \left(\frac{\sqrt{3}}{2}\right)^2 - \left(\frac{1}{2}\right)^2 = \frac{3}{4} - \frac{1}{4} = \frac{2}{4} = \frac{1}{2}$$

$$\operatorname{tg} 60^\circ = \operatorname{tg}(2 \cdot 30^\circ) = \frac{2 \operatorname{tg} 30^\circ}{1 - \operatorname{tg}^2 30^\circ} = \frac{2 \cdot \sqrt{3}/3}{1 - (\sqrt{3}/3)^2} = \frac{2 \cdot \sqrt{3}/3}{1 - 3/9} = \frac{2 \cdot \sqrt{3}/3}{2/3} = \sqrt{3}$$

8. Determina as razóns trigonométricas de 90° a partir das de 45° .

$$\operatorname{sen} 90^\circ = \operatorname{sen}(2 \cdot 45^\circ) = 2 \operatorname{sen} 45^\circ \cos 45^\circ = 2 \cdot \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2} = 1$$

$$\cos 90^\circ = \cos(2 \cdot 45^\circ) = \cos^2 45^\circ - \operatorname{sen}^2 45^\circ = \left(\frac{\sqrt{2}}{2}\right)^2 - \left(\frac{\sqrt{2}}{2}\right)^2 = 0$$

$$\operatorname{tg} 90^\circ = \operatorname{tg}(2 \cdot 45^\circ) = \frac{2 \operatorname{tg} 45^\circ}{1 - \operatorname{tg}^2 45^\circ} = \frac{2 \cdot 1}{1 - 1} \rightarrow \text{No existe.}$$

9. Demostra que:

$$\frac{2 \operatorname{sen} \alpha - \operatorname{sen} 2\alpha}{2 \operatorname{sen} \alpha + \operatorname{sen} 2\alpha} = \frac{1 - \cos \alpha}{1 + \cos \alpha}$$

$$\frac{2 \operatorname{sen} \alpha - \operatorname{sen} 2\alpha}{2 \operatorname{sen} \alpha + \operatorname{sen} 2\alpha} = \frac{2 \operatorname{sen} \alpha - 2 \operatorname{sen} \alpha \cos \alpha}{2 \operatorname{sen} \alpha + 2 \operatorname{sen} \alpha \cos \alpha} = \frac{2 \operatorname{sen} \alpha (1 - \cos \alpha)}{2 \operatorname{sen} \alpha (1 + \cos \alpha)} = \frac{1 - \cos \alpha}{1 + \cos \alpha}$$

Páxina 134

10. Seguindo as indicacións que se dan, demostra detalladamente as fórmulas IV.1, IV.2 e IV.3.

$$\bullet \cos \alpha = \cos \left(2 \cdot \frac{\alpha}{2} \right) = \cos^2 \frac{\alpha}{2} - \operatorname{sen}^2 \frac{\alpha}{2}$$

Por la igualdad fundamental:

$$\cos^2 \frac{\alpha}{2} + \operatorname{sen}^2 \frac{\alpha}{2} = 1 \rightarrow 1 = \cos^2 \frac{\alpha}{2} + \operatorname{sen}^2 \frac{\alpha}{2}$$

De aquí:

a) Sumando ambas igualdades:

$$1 + \cos \alpha = 2 \cos^2 \frac{\alpha}{2} \rightarrow \cos^2 \frac{\alpha}{2} = \frac{1 + \cos \alpha}{2} \rightarrow \cos \frac{\alpha}{2} = \pm \sqrt{\frac{1 + \cos \alpha}{2}}$$

b) Restando las igualdades ($2^a - 1^a$):

$$1 - \cos \alpha = 2 \operatorname{sen}^2 \frac{\alpha}{2} \rightarrow \operatorname{sen}^2 \frac{\alpha}{2} = \frac{1 - \cos \alpha}{2} \rightarrow \operatorname{sen} \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{2}}$$

• Por último:

$$\operatorname{tg} \frac{\alpha}{2} = \frac{\operatorname{sen} \alpha/2}{\cos \alpha/2} = \frac{\pm \sqrt{\frac{1 - \cos \alpha}{2}}}{\pm \sqrt{\frac{1 + \cos \alpha}{2}}} = \sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}}$$

11. Sabendo que $\cos 78^\circ = 0,2$, calcula $\operatorname{sen} 78^\circ$ e $\operatorname{tg} 78^\circ$. Indica as razóns trigonométricas de 39° aplicando as fórmulas do ángulo metade.

$$\bullet \cos 78^\circ = 0,2$$

$$\operatorname{sen} 78^\circ = \sqrt{1 - \cos^2 78^\circ} = \sqrt{1 - 0,2^2} = 0,98$$

$$\operatorname{tg} 78^\circ = \frac{0,98}{0,2} = 4,9$$

$$\bullet \operatorname{sen} 39^\circ = \operatorname{sen} \frac{78^\circ}{2} = \sqrt{\frac{1 - \cos 78^\circ}{2}} = \sqrt{\frac{1 - 0,2}{2}} = 0,63$$

$$\cos 39^\circ = \cos \frac{78^\circ}{2} = \sqrt{\frac{1 + \cos 78^\circ}{2}} = \sqrt{\frac{1 + 0,2}{2}} = 0,77$$

$$\operatorname{tg} 39^\circ = \operatorname{tg} \frac{78^\circ}{2} = \sqrt{\frac{1 - \cos 78^\circ}{1 + \cos 78^\circ}} = \sqrt{\frac{1 - 0,2}{1 + 0,2}} = 0,82$$

12. Determina as razóns trigonométricas de 30° a partir de $\cos 60^\circ = 0,5$.

• $\cos 60^\circ = 0,5$

• $\operatorname{sen} 30^\circ = \operatorname{sen} \frac{60^\circ}{2} = \sqrt{\frac{1 - 0,5}{2}} = 0,5$

$$\cos 30^\circ = \cos \frac{60^\circ}{2} = \sqrt{\frac{1 + 0,5}{2}} = 0,866$$

$$\operatorname{tg} 30^\circ = \operatorname{tg} \frac{60^\circ}{2} = \sqrt{\frac{1 - 0,5}{1 + 0,5}} = 0,577$$

13. Determina as razóns trigonométricas de 45° a partir de $\cos 90^\circ = 0$.

• $\cos 90^\circ = 0$

• $\operatorname{sen} 45^\circ = \operatorname{sen} \frac{90^\circ}{2} = \sqrt{\frac{1 - 0}{2}} = \sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2}$

$$\cos 45^\circ = \cos \frac{90^\circ}{2} = \sqrt{\frac{1 + 0}{2}} = \frac{\sqrt{2}}{2}$$

$$\operatorname{tg} 45^\circ = \operatorname{tg} \frac{90^\circ}{2} = \sqrt{\frac{1 - 0}{1 + 0}} = \sqrt{1} = 1$$

14. Demuestra que $2 \operatorname{tg} \alpha \cdot \operatorname{sen}^2 \frac{\alpha}{2} + \operatorname{sen} \alpha = \operatorname{tg} \alpha$.

$$\begin{aligned} 2 \operatorname{tg} \alpha \cdot \operatorname{sen}^2 \frac{\alpha}{2} + \operatorname{sen} \alpha &= 2 \operatorname{tg} \alpha \cdot \frac{1 - \cos \alpha}{2} + \operatorname{sen} \alpha = \\ &= \frac{\operatorname{sen} \alpha}{\cos \alpha} (1 - \cos \alpha) + \operatorname{sen} \alpha = \operatorname{sen} \alpha \left(\frac{1 - \cos \alpha}{\cos \alpha} + 1 \right) = \\ &= \operatorname{sen} \alpha \left(\frac{1 - \cos \alpha + \cos \alpha}{\cos \alpha} \right) = \operatorname{sen} \alpha \cdot \frac{1}{\cos \alpha} = \\ &= \frac{\operatorname{sen} \alpha}{\cos \alpha} = \operatorname{tg} \alpha \end{aligned}$$

15. Demuestra que $\frac{2 \operatorname{sen} \alpha - \operatorname{sen} 2\alpha}{2 \operatorname{sen} \alpha + \operatorname{sen} 2\alpha} = \operatorname{tg}^2 \frac{\alpha}{2}$.

$$\begin{aligned} \frac{2 \operatorname{sen} \alpha - \operatorname{sen} 2\alpha}{2 \operatorname{sen} \alpha + \operatorname{sen} 2\alpha} &= \frac{2 \operatorname{sen} \alpha - 2 \operatorname{sen} \alpha \cos \alpha}{2 \operatorname{sen} \alpha + 2 \operatorname{sen} \alpha \cos \alpha} = \\ &= \frac{2 \operatorname{sen} \alpha (1 - \cos \alpha)}{2 \operatorname{sen} \alpha (1 + \cos \alpha)} = \frac{1 - \cos \alpha}{1 + \cos \alpha} = \operatorname{tg}^2 \frac{\alpha}{2} \end{aligned}$$

Páxina 135

16. Para demostrar as fórmulas (V.3) e (V.4), dá os seguintes pasos:

- Expresa en función de α y β :

$$\cos(\alpha + \beta) = \dots\dots\dots \quad \cos(\alpha - \beta) = \dots\dots\dots$$

- Suma e resta como fixemos arriba e obterás dúas expresións.

- Substitúe nas expresións anteriores:

$$\left. \begin{array}{l} \alpha + \beta = A \\ \alpha - \beta = B \end{array} \right\} \rightarrow \alpha = \frac{A+B}{2} \quad \beta = \frac{A-B}{2}$$

- $\cos(\alpha + \beta) = \cos \alpha \cos \beta - \operatorname{sen} \alpha \operatorname{sen} \beta$

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \operatorname{sen} \alpha \operatorname{sen} \beta$$

$$\text{Sumando} \rightarrow \cos(\alpha + \beta) + \cos(\alpha - \beta) = 2 \cos \alpha \cos \beta \quad (1)$$

$$\text{Restando} \rightarrow \cos(\alpha + \beta) - \cos(\alpha - \beta) = -2 \operatorname{sen} \alpha \operatorname{sen} \beta \quad (2)$$

- Llamando $\left. \begin{array}{l} \alpha + \beta = A \\ \alpha - \beta = B \end{array} \right\} \rightarrow \alpha = \frac{A+B}{2}, \beta = \frac{A-B}{2}$ (al resolver el sistema)

- Luego, substituyendo en (1) y (2), se obtiene:

$$(1) \rightarrow \cos A + \cos B = 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2}$$

$$(2) \rightarrow \cos A - \cos B = -2 \operatorname{sen} \frac{A+B}{2} \operatorname{sen} \frac{A-B}{2}$$

17. Transforma en produto e calcula:

a) $\operatorname{sen} 75^\circ - \operatorname{sen} 15^\circ$

b) $\cos 75^\circ + \cos 15^\circ$

c) $\cos 75^\circ - \cos 15^\circ$

$$\begin{aligned} \text{a) } \operatorname{sen} 75^\circ - \operatorname{sen} 15^\circ &= 2 \cos \frac{75^\circ + 15^\circ}{2} \operatorname{sen} \frac{75^\circ - 15^\circ}{2} = \\ &= 2 \cos 45^\circ \operatorname{sen} 30^\circ = 2 \cdot \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{\sqrt{2}}{2} \end{aligned}$$

$$\begin{aligned} \text{b) } \cos 75^\circ + \cos 15^\circ &= 2 \cos \frac{75^\circ + 15^\circ}{2} \cos \frac{75^\circ - 15^\circ}{2} = \\ &= 2 \cos 45^\circ \cos 30^\circ = 2 \cdot \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{6}}{2} \end{aligned}$$

$$\begin{aligned} \text{c) } \cos 75^\circ - \cos 15^\circ &= -2 \operatorname{sen} \frac{75^\circ + 15^\circ}{2} \operatorname{sen} \frac{75^\circ - 15^\circ}{2} = \\ &= -2 \operatorname{sen} 45^\circ \operatorname{sen} 30^\circ = -2 \cdot \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = -\frac{\sqrt{2}}{2} \end{aligned}$$

18. Expresa en forma de produto o numerador e o denominador desta fracción e simplifica o resultado:

$$\frac{\underline{\text{sen } 4a + \text{sen } 2a}}{\underline{\text{cos } 4a + \text{cos } 2a}}$$

$$\frac{\text{sen } 4a + \text{sen } 2a}{\text{cos } 4a + \text{cos } 2a} = \frac{2 \text{sen } \frac{4a + 2a}{2} \text{cos } \frac{4a - 2a}{2}}{2 \text{cos } \frac{4a + 2a}{2} \text{cos } \frac{4a - 2a}{2}} = \frac{2 \text{sen } 3a}{2 \text{cos } 3a} = \text{tg } 3a$$

Páxina 137

1. Resolve estas ecuacións:

a) $2 \cos^2 x + \cos x - 1 = 0$

b) $2 \text{sen}^2 x - 1 = 0$

c) $\text{tg}^2 x - \text{tg } x = 0$

d) $2 \text{sen}^2 x + 3 \text{cos } x = 3$

a) $\cos x = \frac{-1 \pm \sqrt{1+8}}{4} = \frac{-1 \pm 3}{4} = \begin{cases} 1/2 \rightarrow x_1 = 60^\circ, x_2 = 300^\circ \\ -1 \rightarrow x_3 = 180^\circ \end{cases}$

Las tres soluciones son válidas (se comprueba en la ecuación inicial).

b) $2 \text{sen}^2 x - 1 = 0 \rightarrow \text{sen}^2 x = \frac{1}{2} \rightarrow \text{sen } x = \pm \frac{1}{\sqrt{2}} = \pm \frac{\sqrt{2}}{2}$

• Si $\text{sen } x = \frac{\sqrt{2}}{2} \rightarrow x_1 = 45^\circ, x_2 = 135^\circ$

• Si $\text{sen } x = -\frac{\sqrt{2}}{2} \rightarrow x_3 = -45^\circ = 315^\circ, x_4 = 225^\circ$

Todas las soluciones son válidas.

c) $\text{tg}^2 x - \text{tg } x = 0 \rightarrow \text{tg } x (\text{tg } x - 1) = 0 \begin{cases} \text{tg } x = 0 \rightarrow x_1 = 0^\circ, x_2 = 180^\circ \\ \text{tg } x = 1 \rightarrow x_3 = 45^\circ, x_4 = 225^\circ \end{cases}$

Todas las soluciones son válidas.

d) $2 \text{sen}^2 x + 3 \text{cos } x = 3 \stackrel{(*)}{\rightarrow} 2(1 - \text{cos}^2 x) + 3 \text{cos } x = 3$

$(*)$ Como $\text{sen}^2 x + \text{cos}^2 x = 1 \rightarrow \text{sen}^2 x = 1 - \text{cos}^2 x$

$2 - 2 \text{cos}^2 x + 3 \text{cos } x = 3 \rightarrow 2 \text{cos}^2 x - 3 \text{cos } x + 1 = 0$

$\text{cos } x = \frac{3 \pm \sqrt{9-8}}{4} = \frac{3 \pm 1}{4} = \begin{cases} 1 \\ 1/2 \end{cases}$

Entonces: • Si $\text{cos } x = 1 \rightarrow x_1 = 0^\circ$

• Si $\text{cos } x = \frac{1}{2} \rightarrow x_2 = 60^\circ, x_3 = -60^\circ = 300^\circ$

Las tres soluciones son válidas.

2. Resolve:

a) $4 \cos 2x + 3 \cos x = 1$

b) $\operatorname{tg} 2x + 2 \cos x = 0$

c) $\sqrt{2} \cos(x/2) - \cos x = 1$

d) $2 \operatorname{sen} x \cos^2 x - 6 \operatorname{sen}^3 x = 0$

$$\begin{aligned} \text{a) } 4 \cos 2x + 3 \cos x = 1 &\rightarrow 4(\cos^2 x - \operatorname{sen}^2 x) + 3 \cos x = 1 \rightarrow \\ &\rightarrow 4(\cos^2 x - (1 - \cos^2 x)) + 3 \cos x = 1 \rightarrow 4(2 \cos^2 x - 1) + 3 \cos x = 1 \rightarrow \\ &\rightarrow 8 \cos^2 x - 4 + 3 \cos x = 1 \Rightarrow 8 \cos^2 x + 3 \cos x - 5 = 0 \rightarrow \\ &\rightarrow \cos x = \frac{-3 \pm \sqrt{9 + 160}}{16} = \frac{-3 \pm 13}{16} = \begin{cases} 10/16 = 5/8 = 0,625 \\ -1 \end{cases} \end{aligned}$$

• Si $\cos x = 0,625 \rightarrow x_1 = 51^\circ 19' 4,13''$, $x_2 = -51^\circ 19' 4,13''$

• Si $\cos x = -1 \rightarrow x_3 = 180^\circ$

Al comprobar las soluciones, las tres son válidas.

b) $\operatorname{tg} 2x + 2 \cos x = 0 \rightarrow \frac{2 \operatorname{tg} x}{1 - \operatorname{tg}^2 x} + 2 \cos x = 0 \rightarrow$

$$\rightarrow \frac{\operatorname{tg} x}{1 - \operatorname{tg}^2 x} + \cos x = 0 \rightarrow \frac{\operatorname{sen} x / \cos x}{1 - (\operatorname{sen}^2 x / \cos^2 x)} + \cos x = 0 \rightarrow$$

$$\rightarrow \frac{\operatorname{sen} x \cos x}{\cos^2 x - \operatorname{sen}^2 x} + \cos x = 0 \rightarrow \operatorname{sen} x \cos x + \cos x (\cos^2 x - \operatorname{sen}^2 x) = 0 \rightarrow$$

$$\rightarrow \cos x (\operatorname{sen} x + \cos^2 x - \operatorname{sen}^2 x) = 0 \rightarrow \cos x (\operatorname{sen} x + 1 - \operatorname{sen}^2 x - \operatorname{sen}^2 x) \rightarrow$$

$$\rightarrow \cos x (1 + \operatorname{sen} x - 2 \operatorname{sen}^2 x) = 0 \rightarrow$$

$$\rightarrow \begin{cases} \cos x = 0 \\ 1 + \operatorname{sen} x - 2 \operatorname{sen}^2 x = 0 \rightarrow \operatorname{sen} x = \frac{-1 \pm \sqrt{1 + 8}}{-4} = \begin{cases} -1/2 \\ 1 \end{cases} \end{cases}$$

• Si $\cos x = 0 \rightarrow x_1 = 90^\circ$, $x_2 = 270^\circ$

• Si $\operatorname{sen} x = -\frac{1}{2} \rightarrow x_3 = 210^\circ$, $x_4 = 330^\circ = -30^\circ$

• Si $\operatorname{sen} x = 1 \rightarrow x_5 = 90^\circ = x_1$

Al comprobar las soluciones, vemos que todas ellas son válidas.

c) $\sqrt{2} \cos \frac{x}{2} - \cos x = 1 \rightarrow \sqrt{2} \sqrt{\frac{1 + \cos x}{2}} - \cos x = 1 \rightarrow$

$$\rightarrow \sqrt{1 + \cos x} - \cos x = 1 \rightarrow \sqrt{1 - \cos x} = 1 + \cos x \rightarrow$$

$$\rightarrow 1 + \cos x = 1 + \cos^2 x + 2 \cos x \rightarrow \cos^2 x + \cos x = 0 \rightarrow \cos x (\cos x + 1) = 0$$

• Si $\cos x = 0 \rightarrow x_1 = 90^\circ$, $x_2 = 270^\circ$

• Si $\cos x = -1 \rightarrow x_3 = 180^\circ$

Al comprobar las soluciones, podemos ver que las únicas válidas son:

$$x_1 = 90^\circ \text{ y } x_3 = 180^\circ$$

$$\begin{aligned} \text{d) } 2 \operatorname{sen} x \cos^2 x - 6 \operatorname{sen}^3 x &= 0 \rightarrow 2 \operatorname{sen} x (\cos^2 x - 3 \operatorname{sen}^2 x) = 0 \rightarrow \\ &\rightarrow 2 \operatorname{sen} x (\cos^2 x + \operatorname{sen}^2 x - 4 \operatorname{sen}^2 x) = 0 \rightarrow 2 \operatorname{sen} x (1 - 4 \operatorname{sen}^2 x) = 0 \end{aligned}$$

$$\bullet \text{ Si } \operatorname{sen} x = 0 \rightarrow x_1 = 0^\circ, x_2 = 180^\circ$$

$$\bullet \text{ Si } \operatorname{sen}^2 x = \frac{1}{4} \rightarrow \operatorname{sen} x = \pm \frac{1}{2} \Rightarrow x_3 = 30^\circ, x_4 = 150^\circ, x_5 = 210^\circ, x_6 = 330^\circ$$

Comprobamos las soluciones y observamos que son válidas todas ellas.

3. Transforma en producto $\operatorname{sen} 3x - \operatorname{sen} x$ e resuelve despois a ecuación $\operatorname{sen} 3x - \operatorname{sen} x = 0$.

$$\operatorname{sen} 3x - \operatorname{sen} x = 0 \rightarrow 2 \cos \frac{3x+x}{2} \operatorname{sen} \frac{3x-x}{2} = 0 \rightarrow 2 \cos 2x \operatorname{sen} x = 0 \rightarrow$$

$$\rightarrow \begin{cases} \cos 2x = 0 \\ \operatorname{sen} x = 0 \end{cases}$$

$$\bullet \text{ Si } \cos 2x = 0 \rightarrow \begin{cases} 2x = 90^\circ & \rightarrow x_1 = 45^\circ \\ 2x = 270^\circ & \rightarrow x_2 = 135^\circ \\ 2x = 90^\circ + 360^\circ & \rightarrow x_3 = 225^\circ \\ 2x = 270^\circ + 360^\circ & \rightarrow x_4 = 315^\circ \end{cases}$$

$$\bullet \text{ Si } \operatorname{sen} x = 0 \Rightarrow x_5 = 0^\circ, x_6 = 180^\circ$$

Comprobamos que las seis soluciones son válidas.

4. Resolve as seguintes ecuacións trigonométricas:

a) $\operatorname{sen}(\pi - x) = \cos\left(\frac{3\pi}{2} - x\right) + \cos \pi$

b) $\operatorname{sen}\left(\frac{\pi}{4} - x\right) + \sqrt{2} \operatorname{sen} x = 0$

$$\left. \begin{array}{l} \text{a) } \operatorname{sen}(\pi - x) = \operatorname{sen} x \\ \cos\left(\frac{3\pi}{2} - x\right) = -\operatorname{sen} x \\ \cos \pi = -1 \end{array} \right\} \text{Entonces, la ecuación queda:}$$

$$\operatorname{sen} x = -\operatorname{sen} x - 1 \rightarrow 2 \operatorname{sen} x = -1 \rightarrow \operatorname{sen} x = \frac{-1}{2}$$

$$\text{Si } \operatorname{sen} x = \frac{-1}{2} \rightarrow x_1 = \frac{7\pi}{6} \text{ rad}, x_2 = \frac{11\pi}{6} \text{ rad}$$

Al comprobar vemos:

$$x_1 = \frac{7\pi}{6} \rightarrow \operatorname{sen}(\pi - x) = \operatorname{sen}\left(\pi - \frac{7\pi}{6}\right) = \operatorname{sen} \frac{-\pi}{6} = \frac{-1}{2}$$

$$\cos\left(\frac{3\pi}{2} - x\right) = \cos\left(\frac{3\pi}{2} - \frac{7\pi}{6}\right) = \cos \frac{2\pi}{6} = \cos \frac{\pi}{3} = \frac{1}{2}$$

Luego la solución es válida, pues:

$$\operatorname{sen}(\pi - x) = \frac{-1}{2} = \cos\left(\frac{3\pi}{2} - x\right) + \cos \pi = \frac{1}{2} + (-1)$$

$$x_2 = \frac{11\pi}{6} \rightarrow \operatorname{sen}(\pi - x) = \operatorname{sen}\left(\pi - \frac{11\pi}{6}\right) = \operatorname{sen}\left(\frac{-5\pi}{6}\right) = -\frac{1}{2}$$

$$\cos\left(\frac{3\pi}{2} - x\right) = \cos\left(\frac{3\pi}{2} - \frac{11\pi}{6}\right) = \cos\left(\frac{-2\pi}{6}\right) = \cos\left(\frac{-\pi}{3}\right) = \frac{1}{2}$$

Luego también es válida esta solución, pues:

$$\operatorname{sen}(\pi - x) = \frac{-1}{2} = \cos\left(\frac{3\pi}{2} - x\right) + \cos \pi = \frac{1}{2} + (-1)$$

Por tanto, las dos soluciones son válidas: $x_1 = \frac{7\pi}{6}$ rad y $x_2 = \frac{11\pi}{6}$ rad

$$\text{b) } \operatorname{sen}\left(\frac{\pi}{4} - x\right) = \operatorname{sen} \frac{\pi}{4} \cos x - \cos \frac{\pi}{4} \operatorname{sen} x = \frac{\sqrt{2}}{2} \cos x - \frac{\sqrt{2}}{2} \operatorname{sen} x$$

Luego la ecuación queda:

$$\frac{\sqrt{2}}{2} \cos x - \frac{\sqrt{2}}{2} \operatorname{sen} x + \sqrt{2} \operatorname{sen} x = 0 \rightarrow \frac{\sqrt{2}}{2} \cos x + \frac{\sqrt{2}}{2} \operatorname{sen} x = 0 \rightarrow$$

$$\rightarrow \cos x + \operatorname{sen} x = 0 \rightarrow \cos x = -\operatorname{sen} x \rightarrow x_1 = \frac{3\pi}{4} \text{ rad, } x_2 = \frac{7\pi}{4} \text{ rad}$$

Comprobamos que ninguna solución vale. Luego la ecuación no tiene solución.

5. Escribe, en radiáns, a expresión xeral de todos os ángulos que verifican:

a) $\operatorname{tg} x = -\sqrt{3}$

b) $\operatorname{sen} x = \cos x$

c) $\operatorname{sen}^2 x = 1$

d) $\operatorname{sen} x = \operatorname{tg} x$

a) $x = 120^\circ + k \cdot 360^\circ$ o bien $x = 300^\circ + k \cdot 360^\circ$

Las dos soluciones quedan recogidas en:

$$x = 120^\circ + k \cdot 180^\circ = \frac{2\pi}{3} + k \pi \text{ rad} = x \text{ con } k \in \mathbb{Z}$$

b) $x = \frac{\pi}{4} + k \pi \text{ rad con } k \in \mathbb{Z}$

$$\left. \begin{array}{l} \text{c) Si } \operatorname{sen} x = 1 \rightarrow x = \frac{\pi}{2} + 2k \pi \text{ rad} \\ \text{Si } \operatorname{sen} x = -1 \rightarrow x = \frac{3\pi}{2} + 2k \pi \text{ rad} \end{array} \right\} \rightarrow x = \frac{\pi}{2} + k \pi \text{ rad con } k \in \mathbb{Z}$$

d) En ese caso debe ocurrir que:

$$\left. \begin{array}{l} \text{O bien } \operatorname{sen} x = 0 \rightarrow x = k \pi \text{ rad} \\ \text{O bien } \cos x = 1 \rightarrow x = 2k \pi \text{ rad} \end{array} \right\} \rightarrow x = k \pi \text{ rad con } k \in \mathbb{Z}$$

EXERCICIOS E PROBLEMAS PROPOSTOS

PARA PRACTICAR

Graos e radiáns

1 Expressa en graos sesaxesimais os seguintes ángulos dados en radiáns:

a) $\frac{\pi}{6}$

b) $\frac{2\pi}{3}$

c) $\frac{4\pi}{3}$

d) $\frac{5\pi}{4}$

e) $\frac{7\pi}{6}$

f) $\frac{9\pi}{2}$

☛ *Faino mentalmente tendo en conta que: π radiáns = 180°.*

a) 30°

b) 120°

c) 240°

d) 225°

e) 210°

f) 810°

2 Expressa en graos sesaxesimais os seguintes ángulos dados en radiáns:

a) 1,5

b) 3,2

c) 5

d) 2,75

a) $\frac{360^\circ}{2\pi} \cdot 1,5 = 85^\circ 56' 37''$

b) $\frac{360^\circ}{2\pi} \cdot 3,2 = 183^\circ 20' 47''$

c) $\frac{360^\circ}{2\pi} \cdot 5 = 286^\circ 28' 44''$

d) $\frac{360^\circ}{2\pi} \cdot 2,75 = 157^\circ 33' 48''$

3 Pasa a radiáns os seguintes ángulos dados en graos. Exprésaos en función de π e en forma decimal.

a) 40°

b) 108°

c) 135°

d) 240°

e) 270°

f) 126°

☛ *Simplifica a expresión que obteñas sen multiplicar por 3,14...*

a) $\frac{40\pi}{180} = \frac{2\pi}{9} \approx 0,7 \text{ rad}$

a) $\frac{2\pi}{360^\circ} \cdot 40^\circ = \frac{2\pi}{9} \approx 0,7 \text{ rad}$

b) $\frac{2\pi}{360^\circ} \cdot 108^\circ = \frac{3\pi}{5} \approx 1,88 \text{ rad}$

c) $\frac{2\pi}{360^\circ} \cdot 135^\circ = \frac{3\pi}{4} \approx 2,36 \text{ rad}$

d) $\frac{2\pi}{360^\circ} \cdot 240^\circ = \frac{4\pi}{3} \approx 4,19 \text{ rad}$

e) $\frac{2\pi}{360^\circ} \cdot 270^\circ = \frac{3\pi}{2} \approx 4,71 \text{ rad}$

f) $\frac{2\pi}{360^\circ} \cdot 126^\circ = \frac{7\pi}{10} \approx 2,2 \text{ rad}$

4 Calcula o resultado das seguintes operacións sen utilizar a calculadora:

$$\text{a) } 5 \cos \frac{\pi}{2} - \cos 0 + 2 \cos \pi - \cos \frac{3\pi}{2} + \cos 2\pi$$

$$\text{b) } 5 \operatorname{tg} \pi + 3 \cos \frac{\pi}{2} - 2 \operatorname{tg} 0 + \operatorname{sen} \frac{3\pi}{2} - 2 \operatorname{sen} 2\pi$$

$$\text{c) } \frac{2}{3} \operatorname{sen} \frac{\pi}{2} - 4 \operatorname{sen} \frac{3\pi}{2} + 3 \operatorname{sen} \pi - \frac{5}{3} \operatorname{sen} \frac{\pi}{2}$$

Comproba o resultado obtido utilizando a calculadora.

$$\text{a) } 5 \cdot 0 - 1 + 2 \cdot (-1) - 0 + 1 = -2$$

$$\text{b) } 5 \cdot 0 + 3 \cdot 0 - 2 \cdot 0 + (-1) - 2 \cdot 0 = -1$$

$$\text{c) } \frac{2}{3} \cdot 1 - 4(-1) + 3 \cdot 0 - \frac{5}{3} \cdot 1 = 3$$

5 Proba que:

$$\text{a) } 4 \operatorname{sen} \frac{\pi}{6} + \sqrt{2} \cos \frac{\pi}{4} + \cos \pi = 2$$

$$\text{b) } 2\sqrt{3} \operatorname{sen} \frac{2\pi}{3} + 4 \operatorname{sen} \frac{\pi}{6} - 2 \operatorname{sen} \frac{\pi}{2} = 3$$

$$\text{a) } 4 \operatorname{sen} \frac{\pi}{6} + \sqrt{2} \cos \frac{\pi}{4} + \cos \pi = 4 \cdot \frac{1}{2} + \sqrt{2} \cdot \frac{\sqrt{2}}{2} + (-1) = 2 + 1 - 1 = 2$$

$$\text{b) } 2\sqrt{3} \operatorname{sen} \frac{2\pi}{3} + 4 \operatorname{sen} \frac{\pi}{6} - 2 \operatorname{sen} \frac{\pi}{2} = 2\sqrt{3} \cdot \frac{\sqrt{3}}{2} + 4 \cdot \frac{1}{2} - 2 \cdot 1 = 3 + 2 - 2 = 3$$

6 Calcula o valor exacto de cada unha destas expresións sen utilizar a calculadora:

$$\text{a) } \operatorname{sen} \frac{\pi}{4} + \operatorname{sen} \frac{\pi}{2} + \operatorname{sen} \pi$$

$$\text{b) } \cos \pi - \cos 0 + \cos \frac{\pi}{2} - \cos \frac{3\pi}{2}$$

$$\text{c) } \operatorname{sen} \frac{2\pi}{3} - \cos \frac{7\pi}{6} + \operatorname{tg} \frac{4\pi}{3} + \operatorname{tg} \frac{11\pi}{6}$$

Comproba os resultados coa calculadora.

$$\text{a) } \frac{\sqrt{2}}{2} + 1 + 0 = \frac{\sqrt{2} + 2}{2}$$

$$\text{b) } -1 - 1 + 0 - 0 = -2$$

$$\text{c) } \frac{\sqrt{3}}{2} - \left(-\frac{\sqrt{3}}{2}\right) + \sqrt{3} + \left(-\frac{\sqrt{3}}{3}\right) = \sqrt{3} \left(\frac{1}{2} + \frac{1}{2} + 1 - \frac{1}{3}\right) = \frac{5\sqrt{3}}{3}$$

7 Calcula o valor exacto destas expresións sen usar a calculadora:

a) $\operatorname{sen} \frac{5\pi}{4} + \operatorname{cos} \frac{3\pi}{4} - \operatorname{sen} \frac{7\pi}{4}$

b) $\operatorname{cos} \frac{5\pi}{3} + \operatorname{tg} \frac{4\pi}{3} - \operatorname{tg} \frac{7\pi}{6}$

c) $\sqrt{3} \operatorname{cos} \frac{\pi}{6} + \operatorname{sen} \frac{\pi}{6} - \sqrt{2} \operatorname{cos} \frac{\pi}{4} - 2\sqrt{3} \operatorname{sen} \frac{\pi}{3}$

Comproba os resultados coa calculadora.

a) $-\frac{\sqrt{2}}{2} + \left(-\frac{\sqrt{2}}{2}\right) - \left(-\frac{\sqrt{2}}{2}\right) = -\frac{\sqrt{2}}{2}$

b) $\frac{1}{2} + \sqrt{3} - \frac{\sqrt{3}}{3} = \frac{1}{2} + \frac{2\sqrt{3}}{3}$

c) $\sqrt{3} \cdot \frac{\sqrt{3}}{2} + \frac{1}{2} - \sqrt{2} \cdot \frac{\sqrt{2}}{2} - 2\sqrt{3} \cdot \frac{\sqrt{3}}{2} = \frac{3}{2} + \frac{1}{2} - 1 - 3 = -2$

8 En cada caso determina, en radiáns, dous valores para o ángulo α tales que:

a) $\operatorname{sen} \alpha = 0,32$

b) $\operatorname{cos} \alpha = 0,58$

c) $\operatorname{tg} \alpha = -1,5$

d) $\operatorname{sen} \alpha = -0,63$

a) $\alpha_1 = 0,33; \alpha_2 = 2,82$

b) $\alpha_1 = 0,95; \alpha_2 = 5,33$

c) $\alpha_1 = -0,98; \alpha_2 = 2,16$

d) $\alpha_1 = -0,68; \alpha_2 = 3,82$

9 Indica, sen pasar a graos, en que cuadrante está cada un dos seguintes ángulos:

a) 2 rad

b) 3,5 rad

c) 5 rad

☛ Ten en conta que:

$$\frac{\pi}{2} \approx 1,57; \quad \pi \approx 3,14; \quad \frac{3\pi}{2} \approx 4,7; \quad 2\pi \approx 6,28$$

a) 2.º cuadrante

b) 3.º cuadrante

c) 4.º cuadrante

Fórmulas trigonométricas

10 Determina as razóns trigonométricas do ángulo de 75° se sabes que $75^\circ = 30^\circ + 45^\circ$.

$$\operatorname{sen} 75^\circ = \operatorname{sen} (30^\circ + 45^\circ) = \operatorname{sen} 30^\circ \operatorname{cos} 45^\circ + \operatorname{cos} 30^\circ \operatorname{sen} 45^\circ =$$

$$= \frac{1}{2} \cdot \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{2} + \sqrt{6}}{4}$$

$$\begin{aligned}\cos 75^\circ &= \cos(30^\circ + 45^\circ) = \cos 30^\circ \cos 45^\circ - \sin 30^\circ \sin 45^\circ = \\ &= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} - \frac{1}{2} \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{6} - \sqrt{2}}{4} \\ \operatorname{tg} 75^\circ &= \operatorname{tg}(30^\circ + 45^\circ) = \frac{\operatorname{tg} 30^\circ + \operatorname{tg} 45^\circ}{1 - \operatorname{tg} 30^\circ \operatorname{tg} 45^\circ} = \frac{\sqrt{3}/3 + 1}{1 - \sqrt{3}/3} = \frac{(\sqrt{3} + 3)/3}{(\sqrt{3} - 3)/3} = \\ &= \frac{3 + \sqrt{3}}{3 - \sqrt{3}} = \frac{(3 + \sqrt{3})^2}{9 - 3} = \frac{9 + 3 + 6\sqrt{3}}{6} = \\ &= \frac{12 + 6\sqrt{3}}{6} = 2 + \sqrt{3}\end{aligned}$$

NOTA: También podemos resolverlo como sigue:

$$\begin{aligned}\operatorname{tg} 75^\circ &= \frac{\operatorname{sen} 75^\circ}{\operatorname{cos} 75^\circ} = \frac{\sqrt{2} + \sqrt{6}}{\sqrt{6} - \sqrt{2}} = \frac{(\sqrt{2} + \sqrt{6})^2}{6 - 2} = \frac{2 + 6 + 2\sqrt{12}}{4} = \\ &= \frac{8 + 4\sqrt{3}}{4} = 2 + \sqrt{3}\end{aligned}$$

- 11** Ao saberes que $\operatorname{sen} x = \frac{3}{5}$ e que $\frac{\pi}{2} < x < \pi$, calcula, sen calcular previamente o valor de x :

a) $\operatorname{sen} 2x$ b) $\operatorname{tg} \frac{x}{2}$ c) $\operatorname{sen} \left(x + \frac{\pi}{6}\right)$
d) $\operatorname{cos} \left(x - \frac{\pi}{3}\right)$ e) $\operatorname{cos} \frac{x}{2}$ f) $\operatorname{tg} \left(x + \frac{\pi}{4}\right)$

• **Calcula $\operatorname{cos} x$ e $\operatorname{tg} x$ e despois aplica as fórmulas.**

$$\operatorname{cos} x = -\sqrt{1 - \operatorname{sen}^2 x} = -\sqrt{1 - \frac{9}{25}} = -\frac{4}{5} \quad (\text{Negativo, por ser del 2.º cuadrante}).$$

$$\operatorname{tg} x = \frac{\operatorname{sen} x}{\operatorname{cos} x} = -\frac{3}{4}$$

$$\text{a) } \operatorname{sen} 2x = 2 \operatorname{sen} x \operatorname{cos} x = 2 \cdot \frac{3}{5} \cdot \left(-\frac{4}{5}\right) = -\frac{24}{25}$$

$$\text{b) } \operatorname{tg} \frac{x}{2} = \sqrt{\frac{1 - \operatorname{cos} x}{1 + \operatorname{cos} x}} = \sqrt{\frac{1 - (-4/5)}{1 + (-4/5)}} = \sqrt{\frac{9/5}{1/5}} = 3$$

Signo positivo, pues si $x \in 2.^\circ$ cuadrante, entonces $\frac{x}{2} \in 1.^\text{er}$ cuadrante.

$$\begin{aligned}\text{c) } \operatorname{sen} \left(x + \frac{\pi}{6}\right) &= \operatorname{sen} x \operatorname{cos} \frac{\pi}{6} + \operatorname{cos} x \operatorname{sen} \frac{\pi}{6} = \\ &= \frac{3}{5} \cdot \frac{\sqrt{3}}{2} + \left(-\frac{4}{5}\right) \cdot \frac{1}{2} = \frac{3\sqrt{3} - 4}{10}\end{aligned}$$

$$d) \cos\left(x - \frac{\pi}{3}\right) = \cos x \cos \frac{\pi}{3} + \operatorname{sen} x \operatorname{sen} \frac{\pi}{3} =$$

$$= \left(-\frac{4}{5}\right) \cdot \frac{1}{2} + \frac{3}{5} \cdot \frac{\sqrt{3}}{2} = \frac{3\sqrt{3} - 4}{10}$$

$$e) \cos \frac{x}{2} \stackrel{(*)}{=} \sqrt{\frac{1 + \cos x}{2}} = \sqrt{\frac{1 - 4/5}{2}} = \sqrt{\frac{1/5}{2}} = \sqrt{\frac{1}{10}} = \frac{\sqrt{10}}{10}$$

(*) Signo positivo, porque $\frac{x}{2} \in 1.\text{er}$ cuadrante.

$$f) \operatorname{tg}\left(x + \frac{\pi}{4}\right) = \frac{\operatorname{tg} x + \operatorname{tg} \pi/4}{1 - \operatorname{tg} x \operatorname{tg} \pi/4} = \frac{-3/4 + 1}{1 - (-3/4) \cdot 1} = \frac{1 - 3/4}{1 + 3/4} = \frac{1}{7}$$

Página 143

12 Indica as razões trigonométricas do ângulo de 15° de dous xeitos, considerando:

$$a) 15^\circ = 45^\circ - 30^\circ$$

$$b) 15^\circ = \frac{30^\circ}{2}$$

$$a) \operatorname{sen} 15^\circ = \operatorname{sen} (45^\circ - 30^\circ) = \operatorname{sen} 45^\circ \cos 30^\circ - \cos 45^\circ \operatorname{sen} 30^\circ =$$

$$= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{\sqrt{6} - \sqrt{2}}{4} = 0,258819$$

$$\cos 15^\circ = \cos (45^\circ - 30^\circ) = \cos 45^\circ \cos 30^\circ + \operatorname{sen} 45^\circ \operatorname{sen} 30^\circ =$$

$$= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{\sqrt{6} + \sqrt{2}}{4} = 0,965926$$

$$\operatorname{tg} 15^\circ = \frac{\operatorname{sen} 15^\circ}{\cos 15^\circ} = \frac{\sqrt{6} - \sqrt{2}}{\sqrt{6} + \sqrt{2}} = \frac{6 + 2 - 2\sqrt{12}}{6 - 2} =$$

$$= \frac{8 - 4\sqrt{3}}{4} = 2 - \sqrt{3} = 0,267949$$

$$b) \operatorname{sen} 15^\circ = \operatorname{sen} \frac{30^\circ}{2} = \sqrt{\frac{1 - \cos 30^\circ}{2}} = \sqrt{\frac{1 - \sqrt{3}/2}{2}} = \sqrt{\frac{2 - \sqrt{3}}{4}} =$$

$$= \frac{\sqrt{2 - \sqrt{3}}}{2} = 0,258819$$

$$\cos 15^\circ = \cos \frac{30^\circ}{2} = \sqrt{\frac{1 + \cos 30^\circ}{2}} = \sqrt{\frac{1 + \sqrt{3}/2}{2}} = \sqrt{\frac{2 + \sqrt{3}}{4}} = 0,9659258$$

$$\operatorname{tg} 15^\circ = \frac{\sqrt{2 - \sqrt{3}}}{\sqrt{2 + \sqrt{3}}} = \frac{0,258819}{0,9659258} = 0,2679491$$

13 Sabendo que $\operatorname{sen} x = 2/3$ e que x é un ángulo do primeiro cuadrante, calcula:

a) $\operatorname{sen} 2x$

b) $\operatorname{tg} \frac{x}{2}$

c) $\cos(30^\circ - x)$

$$\left. \begin{array}{l} \operatorname{sen} x = \frac{2}{3} \\ x \in 1.\text{er cuadrante} \end{array} \right\} \rightarrow \left\{ \begin{array}{l} \cos x, \operatorname{tg} x > 0 \\ \frac{x}{2} \in 1.\text{er cuadrante} \end{array} \right. \rightarrow \left\{ \begin{array}{l} \operatorname{sen} x/2 > 0 \\ \cos x/2 > 0 \\ \operatorname{tg} x/2 > 0 \end{array} \right.$$

$$\bullet \cos x = \sqrt{1 - \operatorname{sen}^2 x} = 1 - \frac{4}{9} = \frac{\sqrt{5}}{3}$$

$$\bullet \operatorname{tg} x = \frac{2/3}{\sqrt{5}/3} = \frac{2\sqrt{5}}{5}$$

$$\text{a) } \operatorname{sen} 2x = 2 \operatorname{sen} x \cos x = 2 \cdot \frac{2}{3} \cdot \frac{\sqrt{5}}{3} = \frac{4\sqrt{5}}{9}$$

$$\begin{aligned} \text{b) } \operatorname{tg} \frac{x}{2} &= \sqrt{\frac{1 - \cos x}{1 + \cos x}} = \sqrt{\frac{1 - 2\sqrt{5}/5}{1 + 2\sqrt{5}/5}} = \sqrt{\frac{5 - 2\sqrt{5}}{5 + 2\sqrt{5}}} = \\ &= \sqrt{\frac{25 + 4 \cdot 5 - 20\sqrt{5}}{25 - 4 \cdot 5}} = \sqrt{\frac{45 - 20\sqrt{5}}{5}} = \sqrt{9 - 4\sqrt{5}} \end{aligned}$$

$$\begin{aligned} \text{c) } \cos(30^\circ - x) &= \cos 30^\circ \cos x + \operatorname{sen} 30^\circ \operatorname{sen} x = \frac{\sqrt{3}}{2} \cdot \frac{2\sqrt{5}}{5} + \frac{1}{2} \cdot \frac{2}{3} = \\ &= \frac{\sqrt{15}}{5} + \frac{1}{3} = \frac{3\sqrt{15} + 5}{15} \end{aligned}$$

14 Se $\operatorname{tg} \alpha = -4/3$ e $90^\circ < \alpha < 180^\circ$, calcula:

a) $\operatorname{sen} \left(\frac{\pi}{2} - \alpha \right)$

b) $\cos \left(180^\circ - \frac{\alpha}{2} \right)$

$$90^\circ < \alpha < 180^\circ \rightarrow \left\{ \begin{array}{l} \operatorname{sen} \alpha > 0 \\ \cos \alpha < 0 \end{array} \right.$$

Además, $\frac{\alpha}{2} \in 1.\text{er cuadrante}$

$$\bullet \operatorname{tg} \alpha = -\frac{4}{3}$$

$$\bullet \frac{1}{\cos^2 \alpha} = \operatorname{tg}^2 \alpha + 1 = \frac{16}{9} + 1 = \frac{25}{9} \rightarrow \cos^2 \alpha = \frac{9}{25} \rightarrow \cos \alpha = -\frac{3}{5}$$

$$\bullet \operatorname{sen} \alpha = \sqrt{1 - \cos^2 \alpha} = \sqrt{1 - \frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$$

$$\text{a) } \operatorname{sen} \left(\frac{\pi}{2} - \alpha \right) = \operatorname{sen} \frac{\pi}{2} \cos \alpha - \cos \frac{\pi}{2} \operatorname{sen} \alpha = 1 \cdot \left(-\frac{3}{5} \right) - 0 \cdot \frac{4}{5} = -\frac{3}{5}$$

$$\begin{aligned}
 \text{b) } \cos\left(180^\circ - \frac{\alpha}{2}\right) &= \cos 180^\circ \cos \frac{\alpha}{2} + \sin 180^\circ \sin \frac{\alpha}{2} = -\cos \frac{\alpha}{2} = \\
 &= -\sqrt{\frac{1 + \cos \alpha}{2}} = -\sqrt{\frac{1 + (-3/5)}{2}} = -\sqrt{\frac{5-3}{10}} = \\
 &= -\sqrt{\frac{2}{10}} = -\sqrt{\frac{1}{5}} = -\frac{\sqrt{5}}{5}
 \end{aligned}$$

15 Sabemos que $\cos x = -\frac{3}{4}$ e $\sin x < 0$.

Sen determinar o valor de x , calcula:

a) $\sin x$

b) $\cos(\pi + x)$

c) $\cos 2x$

d) $\operatorname{tg} \frac{x}{2}$

e) $\sin\left(\frac{\pi}{2} - x\right)$

f) $\cos\left(\pi - \frac{x}{2}\right)$

$$\left. \begin{array}{l} \cos x = -3/4 \\ \sin x < 0 \end{array} \right\} \rightarrow x \in 3.\text{er quadrante} \Rightarrow \frac{x}{2} \in 2.\text{o quadrante}$$

$$\text{a) } \sin x = -\sqrt{1 - \cos^2 x} = -\sqrt{1 - \frac{9}{16}} = -\sqrt{\frac{7}{16}} = -\frac{\sqrt{7}}{4}$$

$$\text{b) } \cos(\pi + x) = \cos \pi \cos x - \sin \pi \sin x = -\cos x = \frac{3}{4}$$

$$\text{c) } \cos 2x = \cos^2 x - \sin^2 x = \frac{9}{16} - \frac{7}{16} = \frac{2}{16} = \frac{1}{8}$$

$$\text{d) } \operatorname{tg} \frac{x}{2} = -\sqrt{\frac{1 - \cos x}{1 + \cos x}} = -\sqrt{\frac{1 + 3/4}{1 - 3/4}} = \sqrt{\frac{7}{1}} = \sqrt{7}$$

$$\text{e) } \sin\left(\frac{\pi}{2} - x\right) = \sin \frac{\pi}{2} \cos x - \cos \frac{\pi}{2} \sin x = \cos x = -\frac{3}{4}$$

$$\text{f) } \cos\left(\pi - \frac{x}{2}\right) = \cos \pi \cos \frac{x}{2} + \sin \pi \sin \frac{x}{2} = -\cos \frac{x}{2} =$$

$$= -\left(-\sqrt{\frac{1 + \cos x}{2}}\right) = \sqrt{\frac{1 - 3/4}{2}} = \sqrt{\frac{1}{8}} = \frac{\sqrt{8}}{8}$$

16 Se $\cos 78^\circ = 0,2$ e $\sin 37^\circ = 0,6$, calcula $\sin 41^\circ$, $\cos 41^\circ$ e $\operatorname{tg} 41^\circ$.

$$41^\circ = 78^\circ - 37^\circ$$

$$\bullet \sin 78^\circ = \sqrt{1 - \cos^2 78^\circ} = \sqrt{1 - 0,2^2} = 0,98$$

$$\bullet \cos 37^\circ = \sqrt{1 - \sin^2 37^\circ} = \sqrt{1 - 0,6^2} = 0,8$$

Ahora, ya podemos calcular:

$$\begin{aligned} \bullet \operatorname{sen} 41^\circ &= \operatorname{sen} (78^\circ - 37^\circ) = \operatorname{sen} 78^\circ \cos 37^\circ - \cos 78^\circ \operatorname{sen} 37^\circ = \\ &= 0,98 \cdot 0,8 - 0,2 \cdot 0,6 = 0,664 \end{aligned}$$

$$\begin{aligned} \bullet \operatorname{cos} 41^\circ &= \operatorname{cos} (78^\circ - 37^\circ) = \operatorname{cos} 78^\circ \cos 37^\circ + \operatorname{sen} 78^\circ \operatorname{sen} 37^\circ = \\ &= 0,2 \cdot 0,8 + 0,98 \cdot 0,6 = 0,748 \end{aligned}$$

$$\bullet \operatorname{tg} 41^\circ = \frac{\operatorname{sen} 41^\circ}{\operatorname{cos} 41^\circ} = \frac{0,664}{0,748} = 0,8877$$

17 Se $\operatorname{tg}(\alpha + \beta) = 4$ e $\operatorname{tg} \alpha = -2$, indica $\operatorname{tg} 2\beta$.

$$\begin{aligned} \operatorname{tg}(\alpha + \beta) &= \frac{\operatorname{tg} \alpha + \operatorname{tg} \beta}{1 - \operatorname{tg} \alpha \operatorname{tg} \beta} \rightarrow 4 = \frac{-2 + \operatorname{tg} \beta}{1 + 2 \operatorname{tg} \beta} \rightarrow \\ &\rightarrow 4 + 8 \operatorname{tg} \beta = -2 + \operatorname{tg} \beta \rightarrow 7 \operatorname{tg} \beta = -6 \rightarrow \\ &\rightarrow \operatorname{tg} \beta = -\frac{6}{7} \end{aligned}$$

Luego:

$$\operatorname{tg} 2\beta = \frac{2 \operatorname{tg} \beta}{1 - \operatorname{tg}^2 \beta} = \frac{2 \cdot (-6/7)}{1 - 36/49} = \frac{-12/7}{13/49} = \frac{-12 \cdot 49}{7 \cdot 13} = -\frac{84}{13}$$

Ecuaciones trigonométricas

18 Resuelve las siguientes ecuaciones:

a) $2 \cos^2 x - \operatorname{sen}^2 x + 1 = 0$ b) $\operatorname{sen}^2 x - \operatorname{sen} x = 0$

c) $2 \cos^2 x - \sqrt{3} \cos x = 0$

• b) e c) son ecuaciones de 2.º grado incompletas.

$$a) \left. \begin{aligned} 2 \cos^2 x - \underbrace{\operatorname{sen}^2 x + 1}_{\cos^2 x} = 0 \end{aligned} \right\} \rightarrow 2 \cos^2 x - \cos^2 x = 0$$

$$\cos^2 x = 0 \rightarrow \cos x = 0 \rightarrow \begin{cases} x_1 = 90^\circ \\ x_2 = 270^\circ \end{cases}$$

Al comprobarlas en la ecuación inicial, las dos soluciones son válidas. Luego:

$$\left. \begin{aligned} x_1 &= 90^\circ + k \cdot 360^\circ = \frac{\pi}{2} + 2k\pi \\ x_2 &= 270^\circ + k \cdot 360^\circ = \frac{3\pi}{2} + 2k\pi \end{aligned} \right\} \text{ con } k \in \mathbb{Z}$$

Lo que podemos expresar como:

$$x = 90^\circ + k \cdot 180^\circ = \frac{\pi}{2} + k\pi \quad \text{con } k \in \mathbb{Z}$$

b) $\text{sen } x (\text{sen } x - 1) = 0 \rightarrow$

$$\rightarrow \begin{cases} \text{sen } x = 0 \rightarrow x_1 = 0^\circ, x_2 = 180^\circ \\ \text{sen } x = 1 \rightarrow x_3 = 90^\circ \end{cases}$$

Comprobando las posibles soluciones, vemos que las tres son válidas. Luego:

$$\left. \begin{aligned} x_1 &= k \cdot 360^\circ = 2k \pi \\ x_2 &= 180^\circ + k \cdot 360^\circ = \pi + 2k \pi \\ x_3 &= 90^\circ + k \cdot 360^\circ = \frac{\pi}{2} + 2k \pi \end{aligned} \right\} \text{ con } k \in \mathbf{Z}$$

O, de otra forma:

$$\left. \begin{aligned} x_1 &= k \pi = k \cdot 180^\circ \\ x_3 &= \frac{\pi}{2} + 2k \pi = 90^\circ + k \cdot 360^\circ \end{aligned} \right\} \text{ con } k \in \mathbf{Z}$$

(x_1 así incluye las soluciones x_1 y x_2 anteriores)

c) $\cos x (2 \cos x - \sqrt{3}) = 0 \rightarrow$

$$\rightarrow \begin{cases} \cos x = 0 \rightarrow x_1 = 90^\circ, x_2 = 270^\circ \\ \cos x = \frac{\sqrt{3}}{2} \rightarrow x_3 = 30^\circ, x_4 = 330^\circ \end{cases}$$

Las cuatro soluciones son válidas. Luego:

$$\left. \begin{aligned} x_1 &= 90^\circ + k \cdot 360^\circ = \frac{\pi}{2} + 2k \pi \\ x_2 &= 270^\circ + k \cdot 360^\circ = \frac{3\pi}{2} + 2k \pi \\ x_3 &= 30^\circ + k \cdot 360^\circ = \frac{\pi}{6} + 2k \pi \\ x_4 &= 330^\circ + k \cdot 360^\circ = \frac{11\pi}{6} + 2k \pi \end{aligned} \right\} \text{ con } k \in \mathbf{Z}$$

NOTA: Obsérvese que las dos primeras soluciones podrían escribirse como una sola de la siguiente forma:

$$x = 90^\circ + k \cdot 180^\circ = \frac{\pi}{2} + k \pi$$

19 Resuelve:

a) $\text{sen}^2 x - \cos^2 x = 1$

b) $\cos^2 x - \text{sen}^2 x = 0$

c) $2 \cos^2 x + \text{sen } x = 1$

d) $3 \text{tg}^2 x - \sqrt{3} \text{tg } x = 0$

a) $(1 - \cos^2 x) - \cos^2 x = 1 \rightarrow 1 - 2 \cos^2 x = 1 \rightarrow \cos^2 x = 0 \rightarrow$

$$\rightarrow \cos x = 0 \rightarrow \begin{cases} x_1 = 90^\circ \\ x_2 = 270^\circ \end{cases}$$

Las dos soluciones son válidas. Luego:

$$\left. \begin{aligned} x_1 &= 90^\circ + k \cdot 360^\circ = \frac{\pi}{2} + 2k\pi \\ x_2 &= 270^\circ + k \cdot 360^\circ = \frac{3\pi}{2} + 2k\pi \end{aligned} \right\} \text{ con } k \in \mathbb{Z}$$

O, lo que es lo mismo:

$$x = 90^\circ + k \cdot 180^\circ = \frac{\pi}{2} + k\pi \quad \text{con } k \in \mathbb{Z}$$

$$\text{b) } (1 - \operatorname{sen}^2 x) - \operatorname{sen}^2 x = 0 \rightarrow 1 - 2 \operatorname{sen}^2 x = 0 \rightarrow$$

$$\rightarrow \operatorname{sen}^2 x = \frac{1}{2} \rightarrow \operatorname{sen} x = \pm \frac{\sqrt{2}}{2}$$

- Si $\operatorname{sen} x = \frac{\sqrt{2}}{2} \rightarrow x_1 = 45^\circ, x_2 = 135^\circ$

- Si $\operatorname{sen} x = -\frac{\sqrt{2}}{2} \rightarrow x_3 = 225^\circ, x_4 = 315^\circ$

Comprobamos que todas las soluciones son válidas. Luego:

$$\left. \begin{aligned} x_1 &= 45^\circ + k \cdot 360^\circ = \frac{\pi}{4} + 2k\pi \\ x_2 &= 135^\circ + k \cdot 360^\circ = \frac{3\pi}{4} + 2k\pi \\ x_3 &= 225^\circ + k \cdot 360^\circ = \frac{5\pi}{4} + 2k\pi \\ x_4 &= 315^\circ + k \cdot 360^\circ = \frac{7\pi}{4} + 2k\pi \end{aligned} \right\} \text{ con } k \in \mathbb{Z}$$

O, lo que es lo mismo:

$$x = 45^\circ + k \cdot 90^\circ = \frac{\pi}{4} + k \cdot \frac{\pi}{2} \quad \text{con } k \in \mathbb{Z}$$

$$\text{c) } 2(1 - \operatorname{sen}^2 x) + \operatorname{sen} x = 1 \rightarrow 2 - 2 \operatorname{sen}^2 x + \operatorname{sen} x = 1 \rightarrow$$

$$\rightarrow 2 \operatorname{sen}^2 x - \operatorname{sen} x - 1 = 0 \rightarrow$$

$$\rightarrow \operatorname{sen} x = \frac{1 \pm \sqrt{1+8}}{4} = \frac{1 \pm 3}{4} = \begin{cases} 1 \rightarrow x_1 = 90^\circ \\ -1/2 \rightarrow x_2 = 210^\circ, x_3 = 330^\circ \end{cases}$$

Las tres soluciones son válidas, es decir:

$$\left. \begin{aligned} x_1 &= 90^\circ + k \cdot 360^\circ = \frac{\pi}{2} + 2k\pi \\ x_2 &= 210^\circ + k \cdot 360^\circ = \frac{7\pi}{6} + 2k\pi \\ x_3 &= 330^\circ + k \cdot 360^\circ = \frac{11\pi}{6} + 2k\pi \end{aligned} \right\} \text{ con } k \in \mathbb{Z}$$

$$d) \operatorname{tg} x (3 \operatorname{tg} x - \sqrt{3}) = 0 \rightarrow$$

$$\rightarrow \begin{cases} \operatorname{tg} x = 0 \rightarrow x_1 = 0^\circ, x_2 = 180^\circ \\ \operatorname{tg} x = \frac{\sqrt{3}}{3} \rightarrow x_3 = 30^\circ, x_4 = 210^\circ \end{cases}$$

Comprobamos las posibles soluciones en la ecuación inicial y vemos que las cuatro son válidas.

Entonces:

$$\left. \begin{aligned} x_1 &= k \cdot 360^\circ = 2k\pi \\ x_2 &= 180^\circ + k \cdot 360^\circ = \pi + 2k\pi \\ x_3 &= 30^\circ + k \cdot 360^\circ = \frac{\pi}{6} + 2k\pi \\ x_4 &= 210^\circ + k \cdot 360^\circ = \frac{7\pi}{6} + 2k\pi \end{aligned} \right\} \text{ con } k \in \mathbb{Z}$$

Lo que podría expresarse con solo dos soluciones que englobaran las cuatro anteriores:

$$x_1 = k \cdot 180^\circ = k\pi \text{ y } x_2 = 30^\circ + k \cdot 180^\circ = \frac{\pi}{6} + k\pi \text{ con } k \in \mathbb{Z}$$

20 Resuelve las siguientes ecuaciones:

$$a) \operatorname{sen} \left(\frac{\pi}{6} - x \right) + \operatorname{cos} \left(\frac{\pi}{3} - x \right) = \frac{1}{2}$$

$$b) \operatorname{sen} 2x - 2 \operatorname{cos}^2 x = 0$$

• *Desenvolve $\operatorname{sen} 2x$ e saca factor común.*

$$c) \operatorname{cos} 2x - 3 \operatorname{sen} x + 1 = 0$$

• *Desenvolve $\operatorname{cos} 2x$ e substitúe $\operatorname{cos}^2 x = 1 - \operatorname{sen}^2 x$.*

$$d) \operatorname{sen} \left(\frac{\pi}{4} + x \right) - \sqrt{2} \operatorname{sen} x = 0$$

$$a) \operatorname{sen} \frac{\pi}{6} \operatorname{cos} x - \operatorname{cos} \frac{\pi}{6} \operatorname{sen} x + \operatorname{cos} \frac{\pi}{3} \operatorname{cos} x + \operatorname{sen} \frac{\pi}{3} \operatorname{sen} x = \frac{1}{2}$$

$$\frac{1}{2} \operatorname{cos} x - \frac{\sqrt{3}}{2} \operatorname{sen} x + \frac{1}{2} \operatorname{cos} x + \frac{\sqrt{3}}{2} \operatorname{sen} x = \frac{1}{2}$$

$$\frac{1}{2} \operatorname{cos} x + \frac{1}{2} \operatorname{cos} x = \frac{1}{2} \rightarrow \operatorname{cos} x = \frac{1}{2} \begin{cases} x_1 = \pi/3 \\ x_2 = 5\pi/3 \end{cases}$$

Comprobamos y vemos que:

$$x_1 \rightarrow \operatorname{sen} \left(\frac{\pi}{6} - \frac{\pi}{3} \right) + \operatorname{cos} \left(\frac{\pi}{3} - \frac{\pi}{3} \right) = \operatorname{sen} \left(-\frac{\pi}{6} \right) + \operatorname{cos} 0 = -\frac{1}{2} + 1 = \frac{1}{2}$$

$$x_2 \rightarrow \operatorname{sen} \left(\frac{\pi}{6} - \frac{5\pi}{3} \right) + \operatorname{cos} \left(\frac{\pi}{3} - \frac{5\pi}{3} \right) = \operatorname{sen} \left(-\frac{3\pi}{2} \right) + \operatorname{cos} \left(-\frac{4\pi}{3} \right) = 1 - \frac{1}{2} = \frac{1}{2}$$

Son válidas las dos soluciones. Luego:

$$\left. \begin{aligned} x_1 &= \frac{\pi}{3} + 2k\pi = 60^\circ + k \cdot 360^\circ \\ x_2 &= \frac{5\pi}{3} + 2k\pi = 300^\circ + k \cdot 360^\circ \end{aligned} \right\} \text{ con } k \in \mathbb{Z}$$

b) $2 \operatorname{sen} x \cos x - 2 \cos^2 x = 0 \rightarrow 2 \cos x (\operatorname{sen} x - \cos x) = 0 \rightarrow$

$$\rightarrow \begin{cases} \cos x = 0 \rightarrow x_1 = 90^\circ, x_2 = 270^\circ \\ \operatorname{sen} x = \cos x \rightarrow x_3 = 45^\circ, x_4 = 225^\circ \end{cases}$$

Comprobamos las soluciones. Todas son válidas:

$$\left. \begin{aligned} x_1 &= 90^\circ + k \cdot 360^\circ = \frac{\pi}{2} + 2k\pi \\ x_2 &= 270^\circ + k \cdot 360^\circ = \frac{3\pi}{2} + 2k\pi \\ x_3 &= 45^\circ + k \cdot 360^\circ = \frac{\pi}{4} + 2k\pi \\ x_4 &= 225^\circ + k \cdot 360^\circ = \frac{5\pi}{4} + 2k\pi \end{aligned} \right\} \text{ con } k \in \mathbb{Z}$$

También podríamos expresarlas como:

$$\left. \begin{aligned} x_1 &= 90^\circ + k \cdot 180^\circ = \frac{\pi}{2} + k\pi \\ x_2 &= 45^\circ + k \cdot 180^\circ = \frac{\pi}{4} + k\pi \end{aligned} \right\} \text{ con } k \in \mathbb{Z}$$

c) $\cos^2 x - \operatorname{sen}^2 x - 3 \operatorname{sen} x + 1 = 0 \rightarrow 1 - \operatorname{sen}^2 x - \operatorname{sen}^2 x - 3 \operatorname{sen} x + 1 = 0 \rightarrow$

$$\rightarrow 1 - 2 \operatorname{sen}^2 x - 3 \operatorname{sen} x + 1 = 0 \rightarrow 2 \operatorname{sen}^2 x + 3 \operatorname{sen} x - 2 = 0 \rightarrow$$

$$\rightarrow \operatorname{sen} x = \frac{-3 \pm \sqrt{9 + 16}}{4} = \frac{-3 \pm 5}{4} = \begin{cases} 1/2 \rightarrow x_1 = 30^\circ, x_2 = 150^\circ \\ -2 \rightarrow \text{¡Imposible!}, \text{ pues } |\operatorname{sen} x| \leq 1 \end{cases}$$

Comprobamos que las dos soluciones son válidas.

Luego:

$$\left. \begin{aligned} x_1 &= 30^\circ + k \cdot 360^\circ = \frac{\pi}{6} + 2k\pi \\ x_2 &= 150^\circ + k \cdot 360^\circ = \frac{5\pi}{6} + 2k\pi \end{aligned} \right\} \text{ con } k \in \mathbb{Z}$$

d) $\operatorname{sen} \frac{\pi}{4} \cos x + \cos \frac{\pi}{4} \operatorname{sen} x - \sqrt{2} \operatorname{sen} x = 0$

$$\frac{\sqrt{2}}{2} \cos x + \frac{\sqrt{2}}{2} \operatorname{sen} x - \sqrt{2} \operatorname{sen} x = 0$$

$$\frac{\sqrt{2}}{2} \cos x - \frac{\sqrt{2}}{2} \operatorname{sen} x = 0 \rightarrow \cos x - \operatorname{sen} x = 0 \rightarrow$$

$$\rightarrow \cos x = \operatorname{sen} x \rightarrow x_1 = \frac{\pi}{4}, x_2 = \frac{5\pi}{4}$$

Al comprobar, podemos ver que ambas soluciones son válidas. Luego:

$$\left. \begin{aligned} x_1 &= \frac{\pi}{4} + 2k\pi = 45^\circ + k \cdot 360^\circ \\ x_2 &= \frac{5\pi}{4} + 2k\pi = 225^\circ + k \cdot 360^\circ \end{aligned} \right\} \text{ con } k \in \mathbb{Z}$$

Podemos agrupar las dos soluciones en:

$$x = \frac{\pi}{4} + k\pi = 45^\circ + k \cdot 180^\circ \quad \text{con } k \in \mathbb{Z}$$

21 Resuelve estas ecuaciones:

a) $4 \operatorname{sen}^2 x \cos^2 x + 2 \cos^2 x - 2 = 0$

• Ao *facer* $\operatorname{sen}^2 x = 1 - \cos^2 x$, resulta unba ecuación bicadrada.

Fai $\cos^2 x = z$ e *comproba* se son válidas as solucións que obtés.

b) $4 \operatorname{sen}^2 x + \operatorname{sen} x \cos x - 3 \cos^2 x = 0$

• *Divide por* $\cos^2 x$ e *obterás* unba ecuación con $\operatorname{tg} x$.

c) $\cos^2 \frac{x}{2} + \cos x - \frac{1}{2} = 0$

d) $\operatorname{tg}^2 \frac{x}{2} + 1 = \cos x$

e) $2 \operatorname{sen}^2 \frac{x}{2} + \cos 2x = 0$

a) $4(1 - \cos^2 x) \cos^2 x + 2 \cos^2 x - 2 = 0$

$$4 \cos^2 x - 4 \cos^4 x + 2 \cos^2 x - 2 = 0$$

$$4 \cos^4 x - 6 \cos^2 x + 2 = 0 \rightarrow 2 \cos^4 x - 3 \cos^2 x + 1 = 0$$

$$\text{Sea } \cos^2 x = z \rightarrow \cos^4 x = z^2$$

Así:

$$2z^2 - 3z + 1 = 0 \rightarrow z = \frac{3 \pm \sqrt{9 - 8}}{4} = \frac{3 \pm 1}{4} \left\langle$$

$$\left\langle \begin{aligned} z_1 = 1 &\rightarrow \cos x = \pm 1 \left\langle \begin{aligned} x_1 &= 0^\circ \\ x_2 &= 180^\circ \end{aligned} \end{aligned}$$

$$z_2 = \frac{1}{2} \rightarrow \cos x = \pm \frac{\sqrt{2}}{2} \left\langle \begin{aligned} x_3 &= 45^\circ, x_4 = 315^\circ \\ x_5 &= 135^\circ, x_6 = 225^\circ \end{aligned}$$

Comprobando las posibles soluciones, vemos que todas son válidas. Por tanto:

$$\left. \begin{aligned} x_1 &= k \cdot 360^\circ = 2k\pi \\ x_2 &= 180^\circ + k \cdot 360^\circ = \pi + 2k\pi \\ x_3 &= 45^\circ + k \cdot 360^\circ = \frac{\pi}{4} + 2k\pi \\ x_4 &= 315^\circ + k \cdot 360^\circ = \frac{5\pi}{4} + 2k\pi \\ x_5 &= 135^\circ + k \cdot 360^\circ = \frac{3\pi}{4} + 2k\pi \\ x_6 &= 225^\circ + k \cdot 360^\circ = \frac{7\pi}{4} + 2k\pi \end{aligned} \right\} \text{ con } k \in \mathbb{Z}$$

O, agrupando las soluciones:

$$\left. \begin{aligned} x_1 &= k \cdot 180^\circ = k\pi \\ x_2 &= 45^\circ + k \cdot 90^\circ = \frac{\pi}{4} + k \frac{\pi}{2} \end{aligned} \right\} \text{ con } k \in \mathbb{Z}$$

b) Dividiendo por $\cos^2 x$:

$$\frac{4 \operatorname{sen}^2 x}{\cos^2 x} + \frac{\operatorname{sen} x \cos x}{\cos^2 x} - \frac{3 \cos^2 x}{\cos^2 x} = 0 \rightarrow 4 \operatorname{tg}^2 x + \operatorname{tg} x - 3 = 0 \rightarrow$$

$$\rightarrow \operatorname{tg} x = \frac{-1 \pm \sqrt{1 + 48}}{8} = \frac{-1 \pm 7}{8} = \begin{cases} \frac{3}{4} \rightarrow \begin{cases} x_1 = 36^\circ 52' 11,6'' \\ x_2 = 216^\circ 52' 11,6'' \end{cases} \\ -1 \rightarrow \begin{cases} x_3 = 135^\circ \\ x_4 = 315^\circ \end{cases} \end{cases}$$

Las cuatro soluciones son válidas:

$$\left. \begin{aligned} x_1 &= 36^\circ 52' 11,6'' + k \cdot 360^\circ \approx \frac{\pi}{5} + 2k\pi \\ x_2 &= 216^\circ 52' 11,6'' + k \cdot 360^\circ \approx \frac{6\pi}{5} + 2k\pi \\ x_3 &= 135^\circ + k \cdot 360^\circ = \frac{3\pi}{4} + 2k\pi \\ x_4 &= 315^\circ + k \cdot 360^\circ = \frac{7\pi}{4} + 2k\pi \end{aligned} \right\} \text{ con } k \in \mathbb{Z}$$

O, lo que es lo mismo:

$$\left. \begin{aligned} x_1 &= 36^\circ 52' 11,6'' + k \cdot 180^\circ \approx \frac{\pi}{5} + k\pi \\ x_2 &= 135^\circ + k \cdot 180^\circ = \frac{3\pi}{4} + k\pi \end{aligned} \right\} \text{ con } k \in \mathbb{Z}$$

$$c) \frac{1 + \cos x}{2} + \cos x - \frac{1}{2} = 0 \rightarrow 1 + \cos x + 2 \cos x - 1 = 0 \rightarrow$$

$$\rightarrow 3 \cos x = 0 \rightarrow \cos x = 0 \rightarrow x_1 = 90^\circ, x_2 = 270^\circ$$

Las dos soluciones son válidas. Luego:

$$\left. \begin{aligned} x_1 = 90^\circ + k \cdot 360^\circ &= \frac{\pi}{2} + 2k\pi \\ x_2 = 270^\circ + k \cdot 360^\circ &= \frac{3\pi}{2} + 2k\pi \end{aligned} \right\} \text{ con } k \in \mathbb{Z}$$

Agrupando las soluciones:

$$x = 90^\circ + k \cdot 180^\circ = \frac{\pi}{2} + k\pi \quad \text{con } k \in \mathbb{Z}$$

$$d) \frac{1 - \cos x}{1 + \cos x} + 1 = \cos x \rightarrow 1 - \cos x + 1 + \cos x = \cos x + \cos^2 x \rightarrow$$

$$\rightarrow 2 = \cos x + \cos^2 x \rightarrow \cos^2 x + \cos x - 2 = 0 \rightarrow$$

$$\rightarrow \cos x = \frac{-1 \pm \sqrt{1+8}}{2} = \frac{-1 \pm 3}{2} \begin{cases} 1 \rightarrow x = 0^\circ \\ -2 \rightarrow \text{¡Imposible!}, \text{ pues } |\cos x| \leq 1 \end{cases}$$

Luego: $x = k \cdot 360^\circ = 2k\pi$ con $k \in \mathbb{Z}$

$$e) 2 \cdot \frac{1 - \cos x}{2} + \cos^2 x - \sin^2 x = 0 \rightarrow$$

$$\rightarrow 1 - \cos x + \cos^2 x - (1 - \cos^2 x) = 0 \rightarrow$$

$$\rightarrow 1 - \cos x + \cos^2 x - 1 + \cos^2 x = 0 \rightarrow 2 \cos^2 x - \cos x = 0 \rightarrow$$

$$\rightarrow \cos x (2 \cos x - 1) = 0 \rightarrow \begin{cases} \cos x = 0 \rightarrow x_1 = 90^\circ, x_2 = 270^\circ \\ \cos x = 1/2 \rightarrow x_3 = 60^\circ, x_4 = 300^\circ \end{cases}$$

Se comprueba que son válidas todas. Por tanto:

$$\left. \begin{aligned} x_1 = 90^\circ + k \cdot 360^\circ &= \frac{\pi}{2} + 2k\pi \\ x_2 = 270^\circ + k \cdot 360^\circ &= \frac{3\pi}{2} + 2k\pi \\ x_3 = 60^\circ + k \cdot 360^\circ &= \frac{\pi}{3} + 2k\pi \\ x_4 = 300^\circ + k \cdot 360^\circ &= \frac{5\pi}{3} + 2k\pi \end{aligned} \right\} \text{ con } k \in \mathbb{Z}$$

Agrupando las soluciones quedaría:

$$\left. \begin{aligned} x_1 = 90^\circ + k \cdot 180^\circ &= \frac{\pi}{2} + k\pi \\ x_2 = 60^\circ + k \cdot 360^\circ &= \frac{\pi}{3} + 2k\pi \\ x_3 = 300^\circ + k \cdot 360^\circ &= \frac{5\pi}{3} + 2k\pi \end{aligned} \right\} \text{ con } k \in \mathbb{Z}$$

Identidades trigonométricas

22 Demuestra que:

$$\frac{\operatorname{sen}(\alpha + \beta)}{\operatorname{sen}(\alpha - \beta)} = \frac{\operatorname{tg} \alpha + \operatorname{tg} \beta}{\operatorname{tg} \alpha - \operatorname{tg} \beta}$$

• *Aplica as fórmulas de $\operatorname{sen}(\alpha + \beta)$ e $\operatorname{sen}(\alpha - \beta)$.*

Divide o numerador e o denominador por $\cos \alpha \cos \beta$ e simplifica.

$$\frac{\operatorname{sen}(\alpha + \beta)}{\operatorname{sen}(\alpha - \beta)} = \frac{\operatorname{sen} \alpha \cos \beta + \cos \alpha \operatorname{sen} \beta}{\operatorname{sen} \alpha \cos \beta - \cos \alpha \operatorname{sen} \beta} \stackrel{(*)}{=}$$

$$= \frac{\frac{\operatorname{sen} \alpha \cos \beta}{\cos \alpha \cos \beta} + \frac{\cos \alpha \operatorname{sen} \beta}{\cos \alpha \cos \beta}}{\frac{\operatorname{sen} \alpha \cos \beta}{\cos \alpha \cos \beta} - \frac{\cos \alpha \operatorname{sen} \beta}{\cos \alpha \cos \beta}} = \frac{\operatorname{tg} \alpha + \operatorname{tg} \beta}{\operatorname{tg} \alpha - \operatorname{tg} \beta}$$

(*) Dividimos numerador y denominador entre $\cos \alpha \cos \beta$.

23 Proba que $2 \operatorname{tg} x \cos^2 \frac{x}{2} - \operatorname{sen} x = \operatorname{tg} x$.

• *Substitúe $\cos^2 \frac{x}{2} = \frac{1 + \cos x}{2}$.*

$$\text{Como } \cos \frac{x}{2} = \pm \sqrt{\frac{1 + \cos x}{2}} \rightarrow \cos^2 \frac{x}{2} = \frac{1 + \cos x}{2}$$

Y substituyendo en la expresión:

$$\begin{aligned} 2 \operatorname{tg} x \cos^2 \frac{x}{2} - \operatorname{sen} x &= 2 \frac{\operatorname{sen} x}{\cos x} \cdot \frac{1 + \cos x}{2} - \operatorname{sen} x = \\ &= \frac{\operatorname{sen} x (1 + \cos x) - \operatorname{sen} x \cos x}{\cos x} \stackrel{(*)}{=} \\ &= \frac{\operatorname{sen} x [1 + \cos x - \cos x]}{\cos x} = \frac{\operatorname{sen} x}{\cos x} = \operatorname{tg} x \end{aligned}$$

(*) Sacando factor común.

24 Demuestra que:

$$\cos \left(x + \frac{\pi}{3} \right) - \cos \left(x + \frac{2\pi}{3} \right) = \cos x$$

• *Desenvolve e substitúe as razóns de $\frac{\pi}{3}$ e $\frac{2\pi}{3}$.*

$$\begin{aligned}
& \cos\left(x + \frac{\pi}{3}\right) - \cos\left(x + \frac{2\pi}{3}\right) = \\
& = \left[\cos x \cos \frac{\pi}{3} - \operatorname{sen} x \operatorname{sen} \frac{\pi}{3}\right] - \left[\cos x \cos \frac{2\pi}{3} - \operatorname{sen} x \operatorname{sen} \frac{2\pi}{3}\right] = \\
& = \left[\cos x \frac{1}{2} - (\operatorname{sen} x) \frac{\sqrt{3}}{2}\right] - \left[\cos x \left(-\frac{1}{2}\right) - (\operatorname{sen} x) \frac{\sqrt{3}}{2}\right] = \\
& = \frac{1}{2} \cos x - \frac{\sqrt{3}}{2} \operatorname{sen} x + \frac{1}{2} \cos x + \frac{\sqrt{3}}{2} \operatorname{sen} x = \cos x
\end{aligned}$$

25 **Demuestra que:**

$$\cos \alpha \cos (\alpha - \beta) + \operatorname{sen} \alpha \operatorname{sen} (\alpha - \beta) = \cos \beta$$

• *Aplica as fórmulas da diferenza de ángulos, simplifica e extrae factor común.*

$$\begin{aligned}
& \cos \alpha \cos (\alpha - \beta) + \operatorname{sen} \alpha \operatorname{sen} (\alpha - \beta) = \\
& = \cos \alpha (\cos \alpha \cos \beta + \operatorname{sen} \alpha \operatorname{sen} \beta) + \operatorname{sen} \alpha (\operatorname{sen} \alpha \cos \beta - \cos \alpha \operatorname{sen} \beta) = \\
& = \cos^2 \alpha \cos \beta + \cos \alpha \operatorname{sen} \alpha \operatorname{sen} \beta + \operatorname{sen}^2 \alpha \cos \beta - \operatorname{sen} \alpha \cos \alpha \operatorname{sen} \beta = \\
& = \cos^2 \alpha \cos \beta + \operatorname{sen}^2 \alpha \cos \beta \stackrel{(*)}{=} \cos \beta (\cos^2 \alpha + \operatorname{sen}^2 \alpha) = \cos \beta \cdot 1 = \cos \beta
\end{aligned}$$

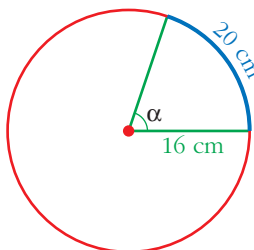
(*) Extraemos factor común.

Páxina 144

PARA RESOLVER

26 Nunha circunferencia de 16 cm de raio, un arco mide 20 cm.

Calcula o ángulo central en graos e en radiáns.



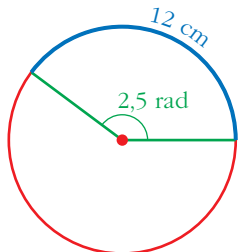
Como a circunferencia completa (100,53 cm) son 2π rad, entonces:

$$\frac{100,53}{20} = \frac{2\pi}{\alpha} \rightarrow \alpha = \frac{20 \cdot 2\pi}{100,53} = 1,25 \text{ rad}$$

$$\alpha = \frac{360^\circ}{2\pi} \cdot 1,25 = 71^\circ 37' 11''$$

- 27** Nunha determinada circunferencia, a un arco de 12 cm de lonxitude corespóndelle un ángulo de 2,5 radiáns.

Cal é o raio desa circunferencia?



$$\frac{2,5 \text{ rad}}{1 \text{ rad}} = \frac{12 \text{ cm}}{R \text{ cm}} \rightarrow R = \frac{12}{2,5} = 4,8 \text{ cm}$$

- 28** Calcula, en radiáns, o ángulo comprendido entre 0 e 2π tal que as súas razóns trigonométricas coincidan coas de $\frac{11\pi}{4}$.

$$0 < \alpha < 2\pi$$

$$\frac{11\pi}{4} = \frac{8\pi + 3\pi}{4} \rightarrow \frac{11\pi}{4} = 2\pi + \frac{3\pi}{4} \Rightarrow \alpha = \frac{3\pi}{4}$$

- 29** Demostra:

$$\frac{\cos(\alpha - \beta)}{\cos(\alpha + \beta)} = \frac{1 + \operatorname{tg} \alpha \operatorname{tg} \beta}{1 - \operatorname{tg} \alpha \operatorname{tg} \beta}$$

$$\frac{\cos(\alpha - \beta)}{\cos(\alpha + \beta)} = \frac{\cos \alpha \cos \beta + \operatorname{sen} \alpha \operatorname{sen} \beta}{\cos \alpha \cos \beta - \operatorname{sen} \alpha \operatorname{sen} \beta} \stackrel{(*)}{=}$$

(*) Dividimos numerador y denominador entre:

$$\cos \alpha \cos \beta$$

$$= \frac{\frac{\cos \alpha \cos \beta}{\cos \alpha \cos \beta} + \frac{\operatorname{sen} \alpha \operatorname{sen} \beta}{\cos \alpha \cos \beta}}{\frac{\cos \alpha \cos \beta}{\cos \alpha \cos \beta} - \frac{\operatorname{sen} \alpha \operatorname{sen} \beta}{\cos \alpha \cos \beta}} = \frac{1 + \operatorname{tg} \alpha \operatorname{tg} \beta}{1 - \operatorname{tg} \alpha \operatorname{tg} \beta}$$

- 30** Simplifica a expresión:

$$\frac{\operatorname{sen} 2\alpha}{1 - \cos^2 \alpha}$$

Calcula o seu valor para $\alpha = \frac{\pi}{4}$.

$$\frac{\operatorname{sen} 2\alpha}{1 - \cos^2 \alpha} = \frac{2 \operatorname{sen} \alpha \cos \alpha}{\operatorname{sen}^2 \alpha} = \frac{2 \cos \alpha}{\operatorname{sen} \alpha}$$

$$\text{Por tanto, si } \alpha = \frac{\pi}{4} \Rightarrow \frac{\operatorname{sen} 2\alpha}{1 - \cos^2 \alpha} = \frac{2 \cos \alpha}{\operatorname{sen} \alpha} = \frac{2 \cdot \left(\frac{\sqrt{2}}{2}\right)}{\frac{\sqrt{2}}{2}} = 2$$

31 Proba que:

$$\frac{2\operatorname{sen} \alpha - \operatorname{sen} 2\alpha}{2\operatorname{sen} \alpha + \operatorname{sen} 2\alpha} = \operatorname{tg}^2 \frac{\alpha}{2}$$

$$\begin{aligned} \frac{2\operatorname{sen} \alpha - \operatorname{sen} 2\alpha}{2\operatorname{sen} \alpha + \operatorname{sen} 2\alpha} &= \frac{2\operatorname{sen} \alpha - 2\operatorname{sen} \alpha \cos \alpha}{2\operatorname{sen} \alpha + 2\operatorname{sen} \alpha \cos \alpha} = \frac{2\operatorname{sen} \alpha (1 - \cos \alpha)}{2\operatorname{sen} \alpha (1 + \cos \alpha)} = \\ &= \frac{1 - \cos \alpha}{1 + \cos \alpha} = \operatorname{tg}^2 \frac{\alpha}{2} \end{aligned}$$

32 Simplifica:

$$\frac{2\cos(45^\circ + \alpha) \cos(45^\circ - \alpha)}{\cos 2\alpha}$$

• Ao desenvolver o numerador obterás unha diferenza de cadrados.

$$\begin{aligned} \frac{2\cos(45^\circ + \alpha) \cos(45^\circ - \alpha)}{\cos 2\alpha} &= \\ &= \frac{2(\cos 45^\circ \cos \alpha - \operatorname{sen} 45^\circ \operatorname{sen} \alpha)(\cos 45^\circ \cos \alpha + \operatorname{sen} 45^\circ \operatorname{sen} \alpha)}{\cos^2 \alpha - \operatorname{sen}^2 \alpha} = \\ &= \frac{2(\cos^2 45^\circ \cos^2 \alpha - \operatorname{sen}^2 45^\circ \operatorname{sen}^2 \alpha)}{\cos^2 \alpha - \operatorname{sen}^2 \alpha} = \\ &= \frac{2 \cdot [(\sqrt{2}/2)^2 \cos^2 \alpha - (\sqrt{2}/2)^2 \operatorname{sen}^2 \alpha]}{\cos^2 \alpha - \operatorname{sen}^2 \alpha} = \frac{2 \cdot 1/2 \cos^2 \alpha - 2 \cdot 1/2 \operatorname{sen}^2 \alpha}{\cos^2 \alpha - \operatorname{sen}^2 \alpha} = \\ &= \frac{\cos^2 \alpha - \operatorname{sen}^2 \alpha}{\cos^2 \alpha - \operatorname{sen}^2 \alpha} = 1 \end{aligned}$$

33 Resolve as seguintes ecuacións:

a) $\cos 2x + 3 \operatorname{sen} x = 2$

b) $\operatorname{tg} 2x \cdot \operatorname{tg} x = 1$

c) $\cos x \cos 2x + 2 \cos^2 x = 0$

d) $2 \operatorname{sen} x = \operatorname{tg} 2x$

e) $\sqrt{3} \operatorname{sen} \frac{x}{2} + \cos x - 1 = 0$

f) $\operatorname{sen} 2x \cos x = 6 \operatorname{sen}^3 x$

g) $\operatorname{tg} \left(\frac{\pi}{4} - x \right) + \operatorname{tg} x = 1$

a) $\cos^2 x - \operatorname{sen}^2 x + 3 \operatorname{sen} x = 2 \rightarrow 1 - \operatorname{sen}^2 x - \operatorname{sen}^2 x + 3 \operatorname{sen} x = 2 \rightarrow$

$\rightarrow 2 \operatorname{sen}^2 x - 3 \operatorname{sen} x + 1 = 0 \rightarrow$

$\rightarrow \operatorname{sen} x = \frac{3 \pm \sqrt{9 - 8}}{4} = \frac{3 \pm 1}{4} \begin{cases} 1 \rightarrow x_1 = 90^\circ \\ 1/2 \rightarrow x_1 = 30^\circ, x_2 = 150^\circ \end{cases}$

Las tres soluciones son válidas:

$$\left. \begin{aligned} x_1 &= 90^\circ + k \cdot 360^\circ = \frac{\pi}{2} + 2k\pi \\ x_2 &= 30^\circ + k \cdot 360^\circ = \frac{\pi}{6} + 2k\pi \\ x_3 &= 150^\circ + k \cdot 360^\circ = \frac{5\pi}{6} + 2k\pi \end{aligned} \right\} \text{ con } k \in \mathbb{Z}$$

$$\begin{aligned} \text{b) } \frac{2 \operatorname{tg} x}{1 - \operatorname{tg}^2 x} \cdot \operatorname{tg} x &= 1 \rightarrow 2 \operatorname{tg}^2 x = 1 - \operatorname{tg}^2 x \rightarrow \operatorname{tg}^2 x = \frac{1}{3} \rightarrow \\ \rightarrow \operatorname{tg} x &= \pm \frac{\sqrt{3}}{3} \rightarrow \begin{cases} x_1 = 30^\circ, & x_2 = 210^\circ \\ x_3 = 150^\circ, & x_4 = 330^\circ \end{cases} \end{aligned}$$

Las cuatro soluciones son válidas:

$$\left. \begin{aligned} x_1 &= 30^\circ + k \cdot 360^\circ = \frac{\pi}{6} + 2k\pi \\ x_2 &= 210^\circ + k \cdot 360^\circ = \frac{7\pi}{6} + 2k\pi \\ x_3 &= 150^\circ + k \cdot 360^\circ = \frac{5\pi}{6} + 2k\pi \\ x_4 &= 330^\circ + k \cdot 360^\circ = \frac{11\pi}{6} + 2k\pi \end{aligned} \right\} \text{ con } k \in \mathbb{Z}$$

Agrupando:

$$\left. \begin{aligned} x_1 &= 30^\circ + k \cdot 180^\circ = \frac{\pi}{6} + k\pi \\ x_2 &= 150^\circ + k \cdot 180^\circ = \frac{5\pi}{6} + k\pi \end{aligned} \right\} \text{ con } k \in \mathbb{Z}$$

$$\text{c) } \cos x (\cos^2 x - \operatorname{sen}^2 x) + 2 \cos^2 x = 0 \rightarrow$$

$$\rightarrow \cos x (\cos^2 x - 1 + \cos^2 x) + 2 \cos^2 x = 0 \rightarrow$$

$$\rightarrow 2 \cos^3 x - \cos x + 2 \cos^2 x = 0 \rightarrow \cos x (2 \cos^2 x + 2 \cos x - 1) = 0 \rightarrow$$

$$\rightarrow \cos x = 0 \rightarrow x_1 = 90^\circ, x_2 = 270^\circ$$

$$\cos x = \frac{-2 \pm \sqrt{4 + 8}}{4} = \frac{-2 \pm 2\sqrt{3}}{4} =$$

$$= \frac{-1 \pm \sqrt{3}}{2} \begin{cases} \approx -1,366 \rightarrow \text{¡Imposible!}, \text{ pues } |\cos x| \leq 1 \\ \approx 0,366 \rightarrow x_3 = 68^\circ 31' 51,1'', x_4 = 291^\circ 28' 8,9'' \end{cases}$$

Las soluciones son todas válidas:

$$\left. \begin{aligned} x_1 &= 90^\circ + k \cdot 360^\circ = \frac{\pi}{2} + 2k\pi \\ x_2 &= 270^\circ + k \cdot 360^\circ = \frac{3\pi}{2} + 2k\pi \\ x_3 &= 68^\circ 31' 51,1'' + k \cdot 360^\circ \approx 0,38\pi + 2k\pi \\ x_4 &= 291^\circ 28' 8,9'' + k \cdot 360^\circ \approx 1,62\pi + 2k\pi \end{aligned} \right\} \text{ con } k \in \mathbb{Z}$$

Agrupadas, serían:

$$\left. \begin{aligned} x_1 &= 90^\circ + k \cdot 180^\circ = \frac{\pi}{2} + k\pi \\ x_2 &= 68^\circ 31' 51,1'' + k \cdot 360^\circ \approx 0,38\pi + 2k\pi \\ x_3 &= 291^\circ 28' 8,9'' + k \cdot 360^\circ \approx 1,62\pi + 2k\pi \end{aligned} \right\} \text{ con } k \in \mathbb{Z}$$

$$d) 2 \operatorname{sen} x = \frac{2 \operatorname{tg} x}{1 - \operatorname{tg}^2 x} \rightarrow 2 \operatorname{sen} x - 2 \operatorname{sen} x \operatorname{tg}^2 x = 2 \operatorname{tg} x \rightarrow$$

$$\rightarrow \operatorname{sen} x - \operatorname{sen} x \frac{\operatorname{sen}^2 x}{\cos^2 x} = \frac{\operatorname{sen} x}{\cos x} \rightarrow$$

$$\rightarrow \operatorname{sen} x \cos^2 x - \operatorname{sen} x \operatorname{sen}^2 x = \operatorname{sen} x \cos x \rightarrow$$

$$\rightarrow \operatorname{sen} x (\cos^2 x - \operatorname{sen}^2 x - \cos x) = 0 \rightarrow$$

$$\rightarrow \operatorname{sen} x (\cos^2 x - 1 + \cos^2 x - \cos x) = 0 \rightarrow$$

$$\rightarrow \begin{cases} \operatorname{sen} x = 0 \rightarrow x_1 = 0^\circ, x_2 = 180^\circ \\ 2 \cos^2 x - \cos x - 1 = 0 \rightarrow \cos x = \frac{1 \pm \sqrt{1+8}}{4} = \end{cases}$$

$$= \begin{cases} 1 \rightarrow x_3 = 0^\circ = x_1 \\ -1/2 \rightarrow x_4 = 240^\circ, x_5 = 120^\circ \end{cases}$$

Las cuatro soluciones son válidas. Luego:

$$\left. \begin{aligned} x_1 &= k \cdot 360^\circ = 2k\pi \\ x_2 &= 180^\circ + k \cdot 360^\circ = \pi + 2k\pi \\ x_4 &= 240^\circ + k \cdot 360^\circ = \frac{4\pi}{3} + 2k\pi \\ x_5 &= 120^\circ + k \cdot 360^\circ = \frac{2\pi}{3} + 2k\pi \end{aligned} \right\} \text{ con } k \in \mathbb{Z}$$

Que, agrupando soluciones, quedaría:

$$\left. \begin{aligned} x_1 &= k \cdot 180^\circ = k\pi \\ x_2 &= 120^\circ + k \cdot 360^\circ = \frac{2\pi}{3} + 2k\pi \\ x_3 &= 240^\circ + k \cdot 360^\circ = \frac{4\pi}{3} + 2k\pi \end{aligned} \right\} \text{ con } k \in \mathbb{Z}$$

$$\begin{aligned}
 \text{e) } \sqrt{3} \sqrt{\frac{1 - \cos x}{2}} + \cos x - 1 &= 0 \rightarrow \frac{3 - 3 \cos x}{2} = (1 - \cos x)^2 \rightarrow \\
 \rightarrow 3 - 3 \cos x &= 2(1 + \cos^2 x - 2 \cos x) \rightarrow 2 \cos^2 x - \cos x - 1 = 0 \rightarrow \\
 \rightarrow \cos x &= \frac{1 \pm \sqrt{1 + 8}}{4} = \frac{1 \pm 3}{4} = \begin{cases} 1 \rightarrow x_1 = 0^\circ \\ -1/2 \rightarrow x_2 = 120^\circ, x_3 = 240^\circ \end{cases}
 \end{aligned}$$

Al comprobar, vemos que las tres soluciones son válidas:

$$\left. \begin{aligned} x_1 &= k \cdot 360^\circ = 2k \pi \\ x_2 &= 120^\circ + k \cdot 360^\circ = \frac{2\pi}{3} + 2k \pi \\ x_3 &= 240^\circ + k \cdot 360^\circ = \frac{4\pi}{3} + 2k \pi \end{aligned} \right\} \text{ con } k \in \mathbb{Z}$$

$$\begin{aligned}
 \text{f) } 2 \operatorname{sen} x \cos x \cos x &= 6 \operatorname{sen}^3 x \rightarrow 2 \operatorname{sen} \cos^2 x = 6 \operatorname{sen}^3 x \rightarrow \\
 \rightarrow 2 \operatorname{sen} x (1 - \operatorname{sen}^2 x) &= 6 \operatorname{sen}^3 x \rightarrow 2 \operatorname{sen} x - 2 \operatorname{sen}^3 x = 6 \operatorname{sen}^3 x \rightarrow \\
 \rightarrow \operatorname{sen} x = 0 &\rightarrow x_1 = 0^\circ, x_2 = 180^\circ
 \end{aligned}$$

$$\operatorname{sen}^2 x = \frac{1}{4} \rightarrow \operatorname{sen} x = \pm \frac{1}{2} \rightarrow \begin{cases} x_3 = 30^\circ, x_4 = 150^\circ \\ x_5 = 210^\circ, x_6 = 330^\circ \end{cases}$$

Comprobamos que todas las soluciones son válidas.

Damos las soluciones agrupando las dos primeras por un lado y el resto por otro:

$$\left. \begin{aligned} x_1 &= k \cdot 180^\circ = k \pi \\ x_2 &= 30^\circ + k \cdot 90^\circ = \frac{\pi}{6} + k \cdot \frac{\pi}{2} \end{aligned} \right\} \text{ con } k \in \mathbb{Z}$$

$$\begin{aligned}
 \text{g) } \frac{\operatorname{tg}(\pi/4) + \operatorname{tg} x}{1 - \operatorname{tg}(\pi/4) \operatorname{tg} x} + \operatorname{tg} x &= 1 \rightarrow \frac{1 + \operatorname{tg} x}{1 - \operatorname{tg} x} + \operatorname{tg} x = 1 \rightarrow \\
 \rightarrow 1 + \operatorname{tg} x + \operatorname{tg} x - \operatorname{tg}^2 x &= 1 - \operatorname{tg} x \rightarrow \operatorname{tg}^2 x - 3 \operatorname{tg} x = 0 \rightarrow \\
 \rightarrow \operatorname{tg} x (\operatorname{tg} x - 3) &= 0 \rightarrow \\
 \rightarrow \begin{cases} \operatorname{tg} x = 0 \rightarrow x_1 = 0^\circ, x_2 = 180^\circ \\ \operatorname{tg} x = 3 \rightarrow x_3 = 71^\circ 33' 54,2'', x_4 = 251^\circ 33' 54,2'' \end{cases}
 \end{aligned}$$

Las cuatro soluciones son válidas:

$$\left. \begin{aligned} x_1 &= k \cdot 360^\circ = 2k \pi \\ x_2 &= 180^\circ + k \cdot 360^\circ = \pi + 2k \pi \\ x_3 &= 71^\circ 33' 54,2'' + k \cdot 360^\circ \approx \frac{2\pi}{5} + 2k \pi \\ x_4 &= 251^\circ 33' 54,2'' + k \cdot 360^\circ \approx \frac{7\pi}{5} + 2k \pi \end{aligned} \right\} \text{ con } k \in \mathbb{Z}$$

O, lo que es lo mismo:

$$\left. \begin{aligned} x_1 &= k \cdot 180^\circ = k \pi \\ x_2 &= 71^\circ 33' 54,2'' + k \cdot 180^\circ \approx \frac{2\pi}{5} + k \pi \end{aligned} \right\} \text{ con } k \in \mathbb{Z}$$

34 Resuelve as seguintes ecuacións:

a) $\text{sen } 3x - \text{sen } x = \text{cos } 2x$

b) $\frac{\text{sen } 5x + \text{sen } 3x}{\text{cos } x + \text{cos } 3x} = 1$

c) $\frac{\text{sen } 3x + \text{sen } x}{\text{cos } 3x + \text{cos } x} = \sqrt{3}$

d) $\text{sen } 3x - \text{cos } 3x = \text{sen } x - \text{cos } x$

• *Transforma as sumas ou diferenzas de senos e cosenos en produtos.*

a) $2 \cos \frac{3x+x}{2} \text{sen } \frac{3x-x}{2} = \text{cos } 2x$

$$2 \cos 2x \text{sen } x = \text{cos } 2x \rightarrow 2 \text{sen } x = 1 \rightarrow \text{sen } x = \frac{1}{2} \rightarrow x_1 = 30^\circ, x_2 = 150^\circ$$

Comprobando, vemos que las dos soluciones son válidas. Luego:

$$\left. \begin{aligned} x_1 &= 30^\circ + k \cdot 360^\circ = \frac{\pi}{6} + 2k \pi \\ x_2 &= 150^\circ + k \cdot 360^\circ = \frac{5\pi}{6} + 2k \pi \end{aligned} \right\} \text{ con } k \in \mathbb{Z}$$

b) $\frac{2 \text{sen } 4x \text{cos } x}{2 \text{cos } 2x \text{cos } x} = 1 \rightarrow \frac{\text{sen } 4x}{\text{cos } 2x} = 1 \rightarrow \frac{\text{sen } (2 \cdot 2x)}{\text{cos } 2x} = 1 \rightarrow$

$$\rightarrow \frac{2 \text{sen } 2x \text{cos } 2x}{\text{cos } 2x} = 1 \rightarrow 2 \text{sen } 2x = 1 \rightarrow \text{sen } 2x = \frac{1}{2} \rightarrow$$

$$\rightarrow \left\{ \begin{aligned} 2x = 30^\circ &\rightarrow x_1 = 15^\circ + k \cdot 360^\circ = \frac{\pi}{12} + 2k \pi \\ 2x = 150^\circ &\rightarrow x_2 = 75^\circ + k \cdot 360^\circ = \frac{5\pi}{12} + 2k \pi \\ 2x = 390^\circ &\rightarrow x_3 = 195^\circ + k \cdot 360^\circ = \frac{13\pi}{12} + 2k \pi \\ 2x = 510^\circ &\rightarrow x_4 = 255^\circ + k \cdot 360^\circ = \frac{17\pi}{12} + 2k \pi \end{aligned} \right\} \text{ con } k \in \mathbb{Z}$$

Al comprobar, vemos que todas las soluciones son válidas.

$$c) \frac{2 \operatorname{sen} 2x \cos x}{-2 \operatorname{sen} 2x \operatorname{sen} x} = \frac{\cos x}{-\operatorname{sen} x} = -\frac{1}{\operatorname{tg} x} = \sqrt{3} \rightarrow \operatorname{tg} x = -\frac{\sqrt{3}}{3} \rightarrow \begin{cases} x_1 = 150^\circ \\ x_2 = 330^\circ \end{cases}$$

Ambas soluciones son válidas. Luego:

$$\left. \begin{aligned} x_1 &= 150^\circ + k \cdot 360^\circ = \frac{5\pi}{6} + 2k\pi \\ x_2 &= 330^\circ + k \cdot 360^\circ = \frac{11\pi}{6} + 2k\pi \end{aligned} \right\} \text{ con } k \in \mathbb{Z}$$

$$d) \operatorname{sen} 3x - \operatorname{sen} x = \cos 3x - \cos x \rightarrow$$

$$\rightarrow 2 \cos 2x \operatorname{sen} x = -2 \operatorname{sen} 2x \operatorname{sen} x \rightarrow (\text{dividimos entre } 2 \operatorname{sen} x)$$

$$\rightarrow \cos 2x = -\operatorname{sen} 2x \rightarrow \frac{\operatorname{sen} 2x}{\cos 2x} = -1 \rightarrow \operatorname{tg} 2x = -1 \rightarrow$$

$$\rightarrow \left\{ \begin{aligned} 2x &= 315^\circ \rightarrow x_1 = 157,5^\circ + k \cdot 360^\circ \\ 2x &= 135^\circ \rightarrow x_2 = 67,5^\circ + k \cdot 360^\circ \\ 2x &= 675^\circ \rightarrow x_3 = 337,5^\circ + k \cdot 360^\circ \\ 2x &= 495^\circ \rightarrow x_4 = 247,5^\circ + k \cdot 360^\circ \end{aligned} \right\} \text{ con } k \in \mathbb{Z}$$

Podemos comprobar que las cuatro soluciones son válidas. Agrupándolas:

$$x = 67,5^\circ + k \cdot 90^\circ \quad \text{con } k \in \mathbb{Z}$$

35 a) Demuestra que: $\operatorname{sen} 3x = 3 \operatorname{sen} x \cos^2 x - \operatorname{sen}^3 x$

b) Resuelve a ecuación $\operatorname{sen} 3x - 2 \operatorname{sen} x = 0$.

a) Fai $\operatorname{sen} 3x = \operatorname{sen} (2x + x)$ e desenvolve.

b) Substitúe $\operatorname{sen} 3x$ polo resultado anterior.

$$a) \operatorname{sen} 3x = \operatorname{sen} (2x + x) = \operatorname{sen} 2x \cos x + \cos 2x \operatorname{sen} x =$$

$$= 2 \operatorname{sen} x \cos x \cos x + (\cos^2 x - \operatorname{sen}^2 x) \operatorname{sen} x =$$

$$= 2 \operatorname{sen} x \cos^2 x + \operatorname{sen} x \cos^2 x - \operatorname{sen}^3 x = 3 \operatorname{sen} x \cos^2 x - \operatorname{sen}^3 x$$

$$b) \operatorname{sen} 3x - 2 \operatorname{sen} x = 0 \rightarrow \text{por el resultado del apartado anterior:}$$

$$3 \operatorname{sen} x \cos^2 x - \operatorname{sen}^3 x - 2 \operatorname{sen} x = 0 \rightarrow 3 \operatorname{sen} x (1 - \operatorname{sen}^2 x) - \operatorname{sen}^3 x - 2 \operatorname{sen} x = 0 \rightarrow$$

$$\rightarrow 3 \operatorname{sen} x - 3 \operatorname{sen}^3 x - \operatorname{sen}^3 x - 2 \operatorname{sen} x = 0 \rightarrow$$

$$\rightarrow 4 \operatorname{sen}^3 x - \operatorname{sen} x = 0 \rightarrow \operatorname{sen} x (4 \operatorname{sen}^2 x - 1) = 0 \rightarrow$$

$$\rightarrow \begin{cases} \operatorname{sen} x = 0 \rightarrow x_1 = 0^\circ, x_2 = 180^\circ \\ \operatorname{sen} x = \pm 1/2 \rightarrow x_3 = 30^\circ, x_4 = 150^\circ, x_5 = 210^\circ, x_6 = 330^\circ \end{cases}$$

Todas las soluciones son válidas y se pueden expresar como:

$$\left. \begin{aligned} x_1 &= k \cdot 180^\circ = k\pi \\ x_2 &= 30^\circ + k \cdot 180^\circ = (\pi/6) + k\pi \\ x_3 &= 150^\circ + k \cdot 180^\circ = (5\pi/6) + k\pi \end{aligned} \right\} \text{ con } k \in \mathbb{Z}$$

36 Demuestra as seguintes igualdades:

a) $\cos(\alpha + \beta) \cdot \cos(\alpha - \beta) = \cos^2 \alpha - \operatorname{sen}^2 \beta$

b) $\operatorname{sen}^2\left(\frac{\alpha + \beta}{2}\right) - \operatorname{sen}^2\left(\frac{\alpha - \beta}{2}\right) = \operatorname{sen} \alpha \cdot \operatorname{sen} \beta$

c) $\cos^2\left(\frac{\alpha - \beta}{2}\right) - \cos^2\left(\frac{\alpha + \beta}{2}\right) = \operatorname{sen} \alpha \cdot \operatorname{sen} \beta$

a) $\cos(\alpha + \beta) \cos(\alpha - \beta) = (\cos \alpha \cos \beta - \operatorname{sen} \alpha \operatorname{sen} \beta) (\cos \alpha \cos \beta + \operatorname{sen} \alpha \operatorname{sen} \beta) =$
 $= \cos^2 \alpha \cos^2 \beta - \operatorname{sen}^2 \alpha \operatorname{sen}^2 \beta =$
 $= \cos^2 \alpha (1 - \operatorname{sen}^2 \beta) - (1 - \cos^2 \alpha) \cdot \operatorname{sen}^2 \beta =$
 $= \cos^2 \alpha - \cos^2 \alpha \operatorname{sen}^2 \beta - \operatorname{sen}^2 \beta + \cos^2 \alpha \operatorname{sen}^2 \beta =$
 $= \cos^2 \alpha - \operatorname{sen}^2 \beta$

b) El primer miembro de la igualdad es una diferencia de cuadrados, luego podemos factorizarlo como una suma por una diferencia:

$$\begin{aligned} & \left[\operatorname{sen}\left(\frac{\alpha + \beta}{2}\right) + \operatorname{sen}\left(\frac{\alpha - \beta}{2}\right) \right] \cdot \left[\operatorname{sen}\left(\frac{\alpha + \beta}{2}\right) - \operatorname{sen}\left(\frac{\alpha - \beta}{2}\right) \right]^{(*)} = \\ & = \left[2 \operatorname{sen} \frac{\alpha}{2} \cos \frac{\beta}{2} \right] \cdot \left[2 \cos \frac{\alpha}{2} \operatorname{sen} \frac{\beta}{2} \right] = \\ & = 4 \sqrt{\frac{1 - \cos \alpha}{2}} \cdot \sqrt{\frac{1 + \cos \beta}{2}} \cdot \sqrt{\frac{1 + \cos \alpha}{2}} \cdot \sqrt{\frac{1 - \cos \beta}{2}} = \\ & = \sqrt{(1 - \cos \alpha)(1 + \cos \beta)(1 + \cos \alpha)(1 - \cos \beta)} = \\ & = \sqrt{(1 - \cos^2 \alpha)(1 - \cos^2 \beta)} = \sqrt{\operatorname{sen}^2 \alpha \cdot \operatorname{sen}^2 \beta} = \operatorname{sen} \alpha \operatorname{sen} \beta \end{aligned}$$

(*) Transformamos la suma y la diferencia en productos, teniendo en cuenta que:

$$\frac{\alpha + \beta}{2} + \frac{\alpha - \beta}{2} = \alpha \quad \text{y} \quad \frac{\alpha + \beta}{2} - \frac{\alpha - \beta}{2} = \beta$$

c) Procedemos de manera análoga al apartado anterior, pero ahora:

$$\frac{\alpha - \beta}{2} + \frac{\alpha + \beta}{2} = \alpha \quad \text{y} \quad \frac{\alpha - \beta}{2} - \frac{\alpha + \beta}{2} = -\beta$$

$$\begin{aligned} & \cos^2\left(\frac{\alpha - \beta}{2}\right) - \cos^2\left(\frac{\alpha + \beta}{2}\right) = \\ & = \left[\cos\left(\frac{\alpha - \beta}{2}\right) + \cos\left(\frac{\alpha + \beta}{2}\right) \right] \cdot \left[\cos\left(\frac{\alpha - \beta}{2}\right) - \cos\left(\frac{\alpha + \beta}{2}\right) \right] = \\ & = \left[2 \cos \frac{\alpha}{2} \cos \frac{-\beta}{2} \right] \cdot \left[-2 \operatorname{sen} \frac{\alpha}{2} \operatorname{sen} \frac{-\beta}{2} \right] = \left[2 \cos \frac{\alpha}{2} \cos \frac{\beta}{2} \right] \cdot \left[2 \operatorname{sen} \frac{\alpha}{2} \operatorname{sen} \frac{\beta}{2} \right] = \end{aligned}$$

$$\begin{aligned}
 &= 4 \sqrt{\frac{1 + \cos \alpha}{2}} \cdot \sqrt{\frac{1 + \cos \beta}{2}} \cdot \sqrt{\frac{1 - \cos \alpha}{2}} \cdot \sqrt{\frac{1 - \cos \beta}{2}} = \\
 &= \sqrt{(1 - \cos^2 \alpha)(1 - \cos^2 \beta)} = \sqrt{\sin^2 \alpha \cdot \sin^2 \beta} = \sin \alpha \sin \beta
 \end{aligned}$$

NOTA: También podríamos haberlo resuelto aplicando el apartado anterior como sigue:

$$\begin{aligned}
 \cos^2 \left(\frac{\alpha - \beta}{2} \right) - \cos^2 \left(\frac{\alpha + \beta}{2} \right) &= 1 - \sin^2 \left(\frac{\alpha - \beta}{2} \right) - 1 + \sin^2 \left(\frac{\alpha + \beta}{2} \right) = \\
 &= \sin^2 \left(\frac{\alpha + \beta}{2} \right) - \sin^2 \left(\frac{\alpha - \beta}{2} \right) \stackrel{(*)}{=} \sin \alpha \sin \beta
 \end{aligned}$$

(*) Por el apartado b).

37 Simplifica a expresión: $\sin \alpha \cdot \cos 2\alpha - \cos \alpha \cdot \sin 2\alpha$

$$\begin{aligned}
 &\sin \alpha (\cos^2 \alpha - \sin^2 \alpha) - \cos \alpha \cdot 2 \sin \alpha \cos \alpha = \\
 &= \sin \alpha \cos^2 \alpha - \sin^3 \alpha - 2 \sin \alpha \cos^2 \alpha = \\
 &= -\sin \alpha \cos^2 \alpha - \sin^3 \alpha = -\sin \alpha (\cos^2 \alpha + \sin^2 \alpha) = -\sin \alpha
 \end{aligned}$$

38 Resolve os sistemas seguintes dando as solucións correspondentes ao primeiro cuadrante:

$$\text{a) } \begin{cases} x + y = 120^\circ \\ \sin x - \sin y = \frac{1}{2} \end{cases}$$

$$\text{b) } \begin{cases} \sin^2 x + \cos^2 y = 1 \\ \cos^2 x - \sin^2 y = 1 \end{cases}$$

• Fai $\cos^2 y = 1 - \sin^2 y$ e $\cos^2 x = 1 - \sin^2 x$.

$$\text{c) } \begin{cases} \sin x + \cos y = 1 \\ x + y = 90^\circ \end{cases}$$

a) De la segunda ecuación:

$$2 \cos \frac{x+y}{2} \sin \frac{x-y}{2} = \frac{1}{2}$$

Como:

$$x + y = 120^\circ \rightarrow 2 \cos 60^\circ \sin \frac{x-y}{2} = \frac{1}{2} \rightarrow 2 \cdot \frac{1}{2} \sin \frac{x-y}{2} = \frac{1}{2} \rightarrow$$

$$\rightarrow \sin \frac{x-y}{2} = \frac{1}{4} \rightarrow \frac{x-y}{2} = 30^\circ \rightarrow x-y = 60^\circ$$

Así: $x + y = 120^\circ$

$$\underline{x - y = 60^\circ}$$

$$2x = 180^\circ \rightarrow x = 90^\circ \rightarrow y = 30^\circ$$

Luego la solución es: $(90^\circ, 30^\circ)$

$$\text{b) Como } \left. \begin{array}{l} \cos^2 y = 1 - \operatorname{sen}^2 y \\ \cos^2 x = 1 - \operatorname{sen}^2 x \end{array} \right\}$$

El sistema queda:

$$\left. \begin{array}{l} \operatorname{sen}^2 x + 1 - \operatorname{sen}^2 y = 1 \\ 1 - \operatorname{sen}^2 x - \operatorname{sen}^2 y = 1 \end{array} \right\} \rightarrow \left. \begin{array}{l} \operatorname{sen}^2 x - \operatorname{sen}^2 y = 0 \\ -\operatorname{sen}^2 x - \operatorname{sen}^2 y = 0 \end{array} \right\}$$

$$(\text{Sumando ambas igualdades}) \rightarrow -2 \operatorname{sen}^2 y = 0 \rightarrow \operatorname{sen} y = 0 \rightarrow y = 0^\circ$$

Sustituyendo en la segunda ecuación (por ejemplo) del sistema inicial, se obtiene:

$$\cos^2 x - 0 = 1 \rightarrow \cos^2 x = 1 = \begin{cases} \cos x = 1 \rightarrow x = 0^\circ \\ \cos x = -1 \rightarrow x = 180^\circ \in 2.^\circ \text{ cuadrante} \end{cases}$$

Luego la solución es: $(0^\circ, 0^\circ)$

$$\text{c) } x + y = 90^\circ \rightarrow \text{complementarios} \rightarrow \operatorname{sen} x = \cos y$$

Sustituyendo en la primera ecuación del sistema:

$$\begin{aligned} \cos y + \cos y = 1 &\rightarrow 2 \cos y = 1 \rightarrow \cos y = \frac{1}{2} \rightarrow y = 60^\circ \rightarrow \\ &\rightarrow x = 90^\circ - y = 90^\circ - 60^\circ = 30^\circ \end{aligned}$$

Luego la solución es: $(30^\circ, 60^\circ)$

39 Xustifica que para calquera ángulo α se verifica:

$$\sqrt{2} \cos \left(\frac{\pi}{4} - \alpha \right) = \operatorname{sen} \alpha + \cos \alpha$$

Desarrollamos la primera parte de la igualdad:

$$\begin{aligned} \sqrt{2} \cdot \cos \left(\frac{\pi}{4} - \alpha \right) &= \sqrt{2} \left(\cos \frac{\pi}{4} \cos \alpha + \operatorname{sen} \frac{\pi}{4} \operatorname{sen} \alpha \right) = \\ &= \sqrt{2} \left(\frac{\sqrt{2}}{2} \cos \alpha + \frac{\sqrt{2}}{2} \operatorname{sen} \alpha \right) = \\ &= \sqrt{2} \cdot \frac{\sqrt{2}}{2} (\cos \alpha + \operatorname{sen} \alpha) = \frac{2}{2} (\cos \alpha + \operatorname{sen} \alpha) = \\ &= \cos \alpha + \operatorname{sen} \alpha \end{aligned}$$

40 Expresa $\operatorname{sen} 4\alpha$ e $\cos 4\alpha$ en función de $\operatorname{sen} \alpha$ e $\cos \alpha$.

- $\operatorname{sen} 4\alpha = \operatorname{sen} (2 \cdot 2\alpha) = 2 \operatorname{sen} \alpha \cos 2\alpha = 2 \cdot 2 \operatorname{sen} \alpha \cos \alpha \cdot (\cos^2 \alpha - \operatorname{sen}^2 \alpha) =$
 $= 4 (\operatorname{sen} \alpha \cos^3 \alpha - \operatorname{sen}^3 \alpha \cos \alpha)$
- $\cos 4\alpha = \cos (2 \cdot 2\alpha) = \cos^2 2\alpha - \operatorname{sen}^2 2\alpha =$
 $= (\cos^2 \alpha - \operatorname{sen}^2 \alpha)^2 - (2 \operatorname{sen} \alpha \cos \alpha)^2 =$
 $= \cos^4 \alpha + \operatorname{sen}^4 \alpha - 2 \cos^2 \alpha \operatorname{sen}^2 \alpha - 4 \operatorname{sen}^2 \alpha \cos^2 \alpha =$
 $= \cos^4 \alpha + \operatorname{sen}^4 \alpha - 6 \operatorname{sen}^2 \alpha \cos^2 \alpha$

Páxina 145

CUESTIÓNS TEÓRICAS

- 41** Que relación existe entre as razóns trigono-métricas dos ángulos que miden $\pi/5$ e $4\pi/5$ radiáns?

$$\frac{\pi}{5} + \frac{4\pi}{5} = \frac{5\pi}{5} = \pi \rightarrow \text{son suplementarios, luego:}$$

$$\text{sen } \frac{\pi}{5} = \text{sen} \left(\pi - \frac{4\pi}{5} \right) = \text{sen } \frac{4\pi}{5}$$

$$\text{cos } \frac{\pi}{5} = -\text{cos } \frac{4\pi}{5}; \quad \text{tg } \frac{\pi}{5} = -\text{tg } \frac{4\pi}{5}$$

- 42** Relaciona estas expresións coas razóns trigonométricas do ángulo α :

a) $\text{sen } (\pi - \alpha)$; $\text{cos } (\pi - \alpha)$; $\text{tg } (\pi - \alpha)$

b) $\text{sen } (\pi + \alpha)$; $\text{cos } (\pi + \alpha)$; $\text{tg } (\pi + \alpha)$

c) $\text{sen } (2\pi - \alpha)$; $\text{cos } (2\pi - \alpha)$; $\text{tg } (2\pi - \alpha)$

a) $\begin{cases} \text{sen } (\pi - \alpha) = \text{sen } \alpha \\ \text{cos } (\pi - \alpha) = -\text{cos } \alpha \end{cases} \rightarrow \text{tg } (\pi - \alpha) = -\text{tg } \alpha$

b) $\begin{cases} \text{sen } (\pi + \alpha) = -\text{sen } \alpha \\ \text{cos } (\pi + \alpha) = -\text{cos } \alpha \end{cases} \rightarrow \text{tg } (\pi + \alpha) = \text{tg } \alpha$

c) $\begin{cases} \text{sen } (2\pi - \alpha) = -\text{sen } \alpha \\ \text{cos } (2\pi - \alpha) = \text{cos } \alpha \end{cases} \rightarrow \text{tg } (2\pi - \alpha) = -\text{tg } \alpha$

- 43** Expresa $A(x)$ en función de $\text{sen } x$ e $\text{cos } x$:

a) $A(x) = \text{sen } (-x) - \text{sen } (\pi - x)$

b) $A(x) = \text{cos } (-x) + \text{cos } (\pi + x)$

c) $A(x) = \text{sen } (\pi + x) + \text{cos } (2\pi - x)$

a) $A(x) = \text{sen } (-x) - \text{sen } (\pi - x) = -\text{sen } x - \text{sen } x = -2 \text{ sen } x$

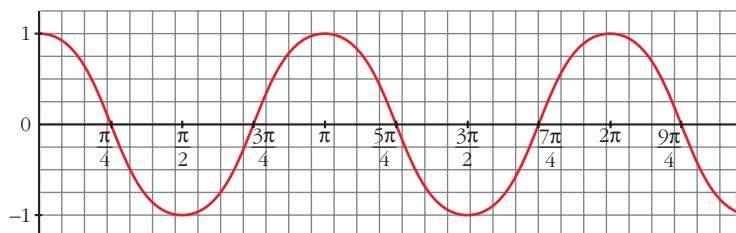
b) $A(x) = \text{cos } (-x) + \text{cos } (\pi + x) = \text{cos } x + (-\text{cos } x) = 0$

c) $A(x) = \text{sen } (\pi + x) + \text{cos } (2\pi - x) = -\text{sen } x + \text{cos } x$

- 44** Fai, coa calculadora, unha táboa de valores da función $y = \cos 2x$, dálle a x valores comprendidos entre 0 e 2π radiáns e representaa graficamente.

x	0	$\frac{\pi}{12}$	$\frac{\pi}{8}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{3\pi}{8}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$	$\frac{7\pi}{12}$	$\frac{5\pi}{8}$	$\frac{2\pi}{3}$
$y = \cos 2x$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$

$\frac{3\pi}{4}$	$\frac{7\pi}{8}$	$\frac{11\pi}{12}$	π	$\frac{5\pi}{4}$	$\frac{7\pi}{8}$	2π
0	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	-1	0	0



PARA AFONDAR

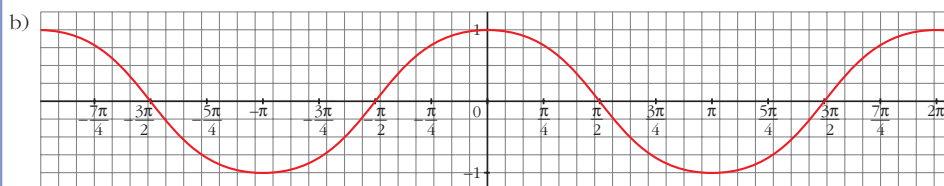
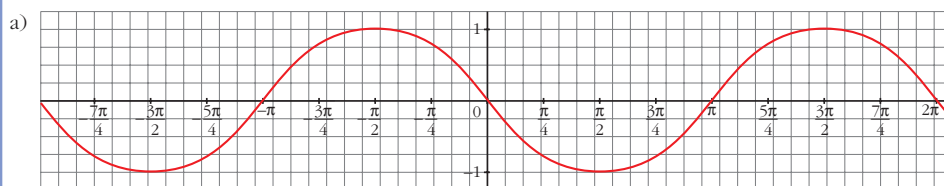
- 45** Representa as funcións:

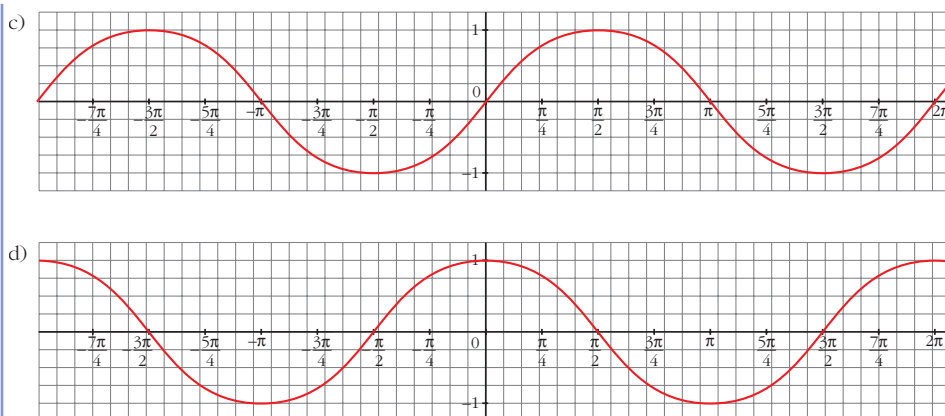
a) $y = \cos\left(x + \frac{\pi}{2}\right)$

b) $y = \text{sen}\left(x + \frac{\pi}{2}\right)$

c) $y = \cos\left(\frac{\pi}{2} - x\right)$

d) $y = \text{sen}\left(\frac{\pi}{2} - x\right)$





46 Resolve os sistemas seguintes e dá as soluções correspondentes ao primeiro quadrante:

a) $\begin{cases} \operatorname{sen} x + \operatorname{sen} y = \sqrt{3} \\ \operatorname{cos} x + \operatorname{cos} y = 1 \end{cases}$ b) $\begin{cases} \operatorname{sen}^2 x + \operatorname{cos}^2 y = 3/4 \\ \operatorname{cos}^2 x - \operatorname{sen}^2 y = 1/4 \end{cases}$ c) $\begin{cases} \operatorname{cos}(x + y) = 1/2 \\ \operatorname{sen}(x - y) = 1/2 \end{cases}$

a) Despejando en la segunda ecuación:

$$\left. \begin{array}{l} \operatorname{cos} x = 1 - \operatorname{cos} y \quad (*) \\ \operatorname{Como} \operatorname{sen} x = \sqrt{1 - \operatorname{cos}^2 x} \end{array} \right\} \text{ entonces:}$$

$$\operatorname{sen} x = \sqrt{1 - (1 - \operatorname{cos} y)^2} = \sqrt{1 - 1 - \operatorname{cos}^2 y + 2 \operatorname{cos} y} = \sqrt{2 \operatorname{cos} y - \operatorname{cos}^2 y}$$

Y, substituyendo en la primera ecuación, se tiene:

$$\begin{aligned} \operatorname{sen} x + \operatorname{sen} y &= \sqrt{3} \rightarrow \sqrt{2 \operatorname{cos} y - \operatorname{cos}^2 y} + \operatorname{sen} y = \sqrt{3} \rightarrow \\ &\rightarrow \operatorname{sen} y = \sqrt{3} - \sqrt{2 \operatorname{cos} y - \operatorname{cos}^2 y} \end{aligned}$$

Elevamos al cuadrado:

$$\operatorname{sen}^2 y = 3 + (2 \operatorname{cos} y - \operatorname{cos}^2 y) - 2\sqrt{3} (2 \operatorname{cos} y - \operatorname{cos}^2 y)$$

$$\operatorname{sen}^2 y + \operatorname{cos}^2 y - 2 \operatorname{cos} y - 3 = -2\sqrt{3} (2 \operatorname{cos} y - \operatorname{cos}^2 y)$$

$$1 - 2 \operatorname{cos} y - 3 = -2\sqrt{3} (2 \operatorname{cos} y - \operatorname{cos}^2 y)$$

$$-2(1 + \operatorname{cos} y) = -2\sqrt{3} (2 \operatorname{cos} y - \operatorname{cos}^2 y)$$

Simplificamos y volvemos a elevar al cuadrado:

$$(1 + \operatorname{cos} y)^2 = 3(2 \operatorname{cos} y - \operatorname{cos}^2 y) \rightarrow$$

$$\rightarrow 1 + \operatorname{cos}^2 y + 2 \operatorname{cos} y = 6 \operatorname{cos} y - 3 \operatorname{cos}^2 y \rightarrow$$

$$\rightarrow 4 \operatorname{cos}^2 y - 4 \operatorname{cos} y + 1 = 0 \rightarrow \operatorname{cos} y = \frac{4 \pm \sqrt{16 - 16}}{8} = \frac{1}{2} \rightarrow y = 60^\circ$$

Sustituyendo en (*), se tiene:

$$\cos x = 1 - \frac{1}{2} = \frac{1}{2} \rightarrow x = 60^\circ$$

$$\left. \begin{array}{l} \text{b) } \operatorname{sen}^2 x + \cos^2 y = \frac{3}{4} \\ \operatorname{cos}^2 x - \operatorname{sen}^2 y = \frac{1}{4} \end{array} \right\} \text{Sumando:}$$

$$\begin{aligned} \operatorname{sen}^2 x + \cos^2 x + \cos^2 y - \operatorname{sen}^2 y &= 1 \rightarrow 1 + \cos^2 y - \operatorname{sen}^2 y = 1 \rightarrow \\ \rightarrow 2 \cos^2 y &= 1 \rightarrow \cos^2 y = \frac{1}{2} \rightarrow \cos y = \frac{\sqrt{2}}{2} \rightarrow y = 45^\circ \end{aligned}$$

(Solo consideramos las soluciones del primer cuadrante).

Sustituyendo en la primera ecuación:

$$\begin{aligned} \operatorname{sen}^2 x + \cos^2 y &= \frac{3}{4} \rightarrow \operatorname{sen}^2 x + \frac{1}{2} = \frac{3}{4} \rightarrow \\ \rightarrow \operatorname{sen}^2 x &= \frac{3}{4} - \frac{1}{2} \rightarrow \operatorname{sen}^2 x = \frac{1}{4} \rightarrow \operatorname{sen} x = \pm \frac{1}{2} \end{aligned}$$

Nos quedamos con la solución positiva, por tratarse del primer cuadrante. Así:

$$\operatorname{sen} x = \frac{1}{2} \rightarrow x = 30^\circ$$

Luego la solución es: $(30^\circ, 45^\circ)$

$$\text{c) Como } x, y \in 1.^{\text{er}} \text{ cuadrante} \left. \begin{array}{l} \text{y además } \operatorname{cos}(x+y) > 0 \\ \operatorname{sen}(x-y) > 0 \end{array} \right\} \rightarrow \left\{ \begin{array}{l} x+y \in 1.^{\text{er}} \text{ cuadrante} \\ x-y \in 1.^{\text{er}} \text{ cuadrante} \end{array} \right.$$

Teniendo esto en cuenta:

$$\operatorname{cos}(x+y) = \frac{1}{2} \rightarrow x+y = 60^\circ$$

$$\operatorname{sen}(x-y) = \frac{1}{2} \rightarrow x-y = 30^\circ \text{ (Sumamos ambas ecuaciones)}$$

$$2x = 90^\circ \rightarrow x = 45^\circ$$

Sustituyendo en la primera ecuación y despejando:

$$y = 60^\circ - x = 60^\circ - 45^\circ = 15^\circ$$

La solución es, por tanto: $(45^\circ, 15^\circ)$

47 Demuestra que:

$$\text{a) } \operatorname{sen} x = \frac{2 \operatorname{tg} x/2}{1 + \operatorname{tg}^2 x/2} \quad \text{b) } \cos x = \frac{1 - \operatorname{tg}^2 x/2}{1 + \operatorname{tg}^2 x/2} \quad \text{c) } \operatorname{tg} x = \frac{2 \operatorname{tg} x/2}{1 - \operatorname{tg}^2 x/2}$$

a) Desarrollamos y operamos en el segundo miembro de la igualdad:

$$\begin{aligned} \frac{2 \operatorname{tg} (x/2)}{1 + \operatorname{tg}^2 (x/2)} &= \frac{2 \sqrt{\frac{1 - \cos x}{1 + \cos x}}}{1 + \frac{1 - \cos x}{1 + \cos x}} = \frac{2 \sqrt{\frac{1 - \cos x}{1 + \cos x}}}{\frac{1 + \cos x + 1 - \cos x}{1 + \cos x}} = \\ &= \frac{2 \sqrt{\frac{1 - \cos x}{1 + \cos x}}}{\frac{2}{1 + \cos x}} = (1 + \cos x) \sqrt{\frac{1 - \cos x}{1 + \cos x}} = \\ &= \sqrt{(1 + \cos x)^2 \frac{1 - \cos x}{1 + \cos x}} = \sqrt{(1 + \cos x)(1 - \cos x)} = \\ &= \sqrt{1 - \cos^2 x} = \sqrt{\operatorname{sen}^2 x} = \operatorname{sen} x \end{aligned}$$

$$\text{b) } \frac{1 - \operatorname{tg}^2 (x/2)}{1 + \operatorname{tg}^2 (x/2)} = \frac{1 - \frac{1 - \cos x}{1 + \cos x}}{1 + \frac{1 - \cos x}{1 + \cos x}} = \frac{\frac{1 + \cos x - 1 + \cos x}{1 + \cos x}}{\frac{1 + \cos x + 1 - \cos x}{1 + \cos x}} = \frac{2 \cos x}{2} = \cos x$$

$$\begin{aligned} \text{c) } \frac{2 \operatorname{tg} (x/2)}{1 - \operatorname{tg}^2 (x/2)} &= \frac{2 \sqrt{\frac{1 - \cos x}{1 + \cos x}}}{1 - \frac{1 - \cos x}{1 + \cos x}} = \frac{2 \sqrt{\frac{1 - \cos x}{1 + \cos x}}}{\frac{1 + \cos x - 1 + \cos x}{1 + \cos x}} = \\ &= \frac{2 \sqrt{\frac{1 - \cos x}{1 + \cos x}}}{\frac{2 \cos x}{1 + \cos x}} = \frac{1 + \cos x}{\cos x} \sqrt{\frac{1 - \cos x}{1 + \cos x}} = \\ &= \frac{1}{\cos x} \cdot \sqrt{(1 + \cos x)^2 \frac{1 - \cos x}{1 + \cos x}} = \\ &= \frac{1}{\cos x} \sqrt{(1 + \cos x)(1 - \cos x)} = \frac{1}{\cos x} \sqrt{1 - \cos^2 x} \\ &= \frac{1}{\cos x} \cdot \sqrt{\operatorname{sen}^2 x} = \frac{1}{\cos x} \cdot \operatorname{sen} x = \operatorname{tg} x \end{aligned}$$

AUTOAVALIACIÓN

1. Expresa en graos: $\frac{3\pi}{4}$ rad, $\frac{5\pi}{2}$ rad, 2 rad.

$$\frac{3\pi}{4} \text{ rad} = 135^\circ$$

$$\frac{5\pi}{2} \text{ rad} = 450^\circ$$

$$2 \text{ rad} = 114^\circ 35' 30''$$

2. Expresa en radiáns e dá o resultado en función de π e como número decimal:

a) 60°

b) 225°

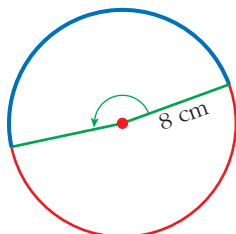
c) 330°

a) $60^\circ = \frac{\pi}{3} \text{ rad} = 1,05 \text{ rad}$

b) $225^\circ = \frac{5\pi}{4} \text{ rad} = 3,93 \text{ rad}$

c) $330^\circ = \frac{11\pi}{6} \text{ rad} = 5,76 \text{ rad}$

3. Nunha circunferencia de 16 cm de diámetro debuxamos un ángulo de 3 rad. Que lonxitude terá o arco correspondente?



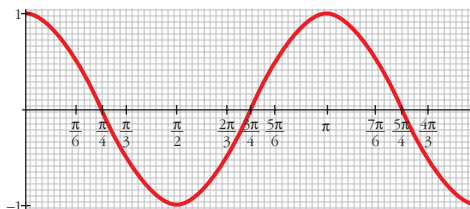
$$l = 8 \cdot 3 = 24 \text{ cm}$$

4. Asocia a esta gráfica unha das seguintes expresións e di cal é o seu período:

a) $y = \cos x$

b) $y = \cos 2x$

c) $y = 2\cos x$



Completa estes puntos para que pertencen á gráfica: $(5\pi/6, \dots)$, $(4\pi/3, \dots)$, $(-\pi/4, \dots)$.

La gráfica corresponde a la b) $y = \cos 2x$. Su periodo es π .

$$\left(\frac{5\pi}{6}, \dots\right) \rightarrow y = \cos 2 \cdot \frac{5\pi}{6} = \frac{1}{2} \rightarrow \left(\frac{5\pi}{6}, \frac{1}{2}\right)$$

$$\left(\frac{4\pi}{3}, \dots\right) \rightarrow y = \cos 2 \cdot \frac{4\pi}{3} = -\frac{1}{2} \rightarrow \left(\frac{4\pi}{3}, -\frac{1}{2}\right)$$

$$\left(\frac{\pi}{4}, \dots\right) \rightarrow y = \cos 2 \cdot \left(\frac{\pi}{4}\right) = 0 \rightarrow \left(-\frac{\pi}{4}, 0\right)$$

5. Se $\cos \alpha = -\frac{1}{4}$ e $\alpha < \pi$, determina:

a) $\text{sen } 2\alpha$ b) $\cos(\pi + \alpha)$ c) $\text{tg } \frac{\alpha}{2}$ d) $\text{sen}\left(\frac{\pi}{6} - \alpha\right)$

$$\cos \alpha = -\frac{1}{4} \quad \alpha < \pi \rightarrow \text{sen}^2 \alpha = 1 - \left(-\frac{1}{4}\right)^2 = \frac{15}{16} \rightarrow \text{sen } \alpha = \frac{\sqrt{15}}{4}$$

$$\text{a) } \text{sen } 2\alpha = 2 \text{sen } \alpha \cos \alpha = 2 \left(-\frac{1}{4}\right) \left(\frac{\sqrt{15}}{4}\right) = -\frac{\sqrt{15}}{8}$$

$$\text{b) } \cos(\pi + \alpha) = -\cos \alpha = \frac{1}{4}$$

$$\text{c) } \text{tg } \frac{\alpha}{2} = \sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}} = \sqrt{\frac{1 - (-1/4)}{1 + (-1/4)}} = \sqrt{\frac{5}{3}}$$

$$\begin{aligned} \text{d) } \text{sen}\left(\frac{\pi}{6} - \alpha\right) &= \text{sen } \frac{\pi}{6} \cos \alpha - \cos \frac{\pi}{6} \text{sen } \alpha = \frac{1}{2} \left(-\frac{1}{4}\right) - \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{15}}{4} = \\ &= -\frac{1}{8} - \frac{\sqrt{45}}{8} = \frac{-1 - 3\sqrt{5}}{8} \end{aligned}$$

6. Demuestra cada unha destas igualdades:

$$\text{a) } \text{tg } 2\alpha = \frac{2 \text{tg } \alpha}{1 - \text{tg}^2 \alpha}$$

$$\text{b) } \text{sen}(\alpha + \beta) \cdot \text{sen}(\alpha - \beta) = \text{sen}^2 \alpha - \text{sen}^2 \beta$$

$$\text{a) } \text{tg } 2\alpha = \frac{\text{sen } 2\alpha}{\cos 2\alpha} = \frac{2 \text{sen } \alpha \cos \alpha}{\cos^2 \alpha - \text{sen}^2 \alpha} = \frac{\frac{2 \text{sen } \alpha \cos \alpha}{\cos^2 \alpha}}{1 - \frac{\text{sen}^2 \alpha}{\cos^2 \alpha}} = \frac{2 \text{tg } \alpha}{1 - \text{tg}^2 \alpha}$$

$$\begin{aligned} \text{b) } \text{sen}(\alpha + \beta) \cdot \text{sen}(\alpha - \beta) &= \\ &= (\text{sen } \alpha \cos \beta + \cos \alpha \text{sen } \beta) (\text{sen } \alpha \cos \beta - \cos \alpha \text{sen } \beta) = \\ &= \text{sen}^2 \alpha \cos^2 \beta - \cos^2 \alpha \text{sen}^2 \beta = \text{sen}^2 \alpha (1 - \text{sen}^2 \beta) - (1 - \text{sen}^2 \alpha) \text{sen}^2 \beta = \\ &= \text{sen}^2 \alpha - \text{sen}^2 \alpha \text{sen}^2 \beta - \text{sen}^2 \beta + \text{sen}^2 \alpha \text{sen}^2 \beta = \text{sen}^2 \alpha - \text{sen}^2 \beta \end{aligned}$$

7. Resolve:

$$\text{a) } \cos 2x - \cos\left(\frac{\pi}{2} + x\right) = 1$$

$$\text{b) } 2\operatorname{tg} x \cos^2 \frac{x}{2} - \operatorname{sen} x = 1$$

$$\text{a) } \cos 2x - \cos\left(\frac{\pi}{2} + x\right) = 1$$

$$\cos^2 x - \operatorname{sen}^2 x - (-\operatorname{sen} x) = 1 \rightarrow 1 - \operatorname{sen}^2 x - \operatorname{sen}^2 x + \operatorname{sen} x - 1 = 0$$

$$-2\operatorname{sen}^2 x + \operatorname{sen} x = 0 \rightarrow \operatorname{sen} x(-2\operatorname{sen} x + 1) = 0 \begin{cases} \operatorname{sen} x = 0 & \begin{cases} x = 0^\circ \\ x = 180^\circ \end{cases} \\ \operatorname{sen} x = \frac{1}{2} & \begin{cases} x = 30^\circ \\ x = 150^\circ \end{cases} \end{cases}$$

Soluciones:

$$x_1 = 360^\circ k; \quad x_2 = 180^\circ + 360^\circ k; \quad x_3 = 30^\circ + 360^\circ k; \quad x_4 = 150^\circ + 360^\circ k, \quad \text{con } k \in \mathbb{Z}$$

$$\text{b) } 2\operatorname{tg} x \cos^2 \frac{x}{2} - \operatorname{sen} x = 1 \rightarrow 2\operatorname{tg} x \frac{1 + \cos x}{2} - \operatorname{sen} x = 1 \rightarrow$$

$$\rightarrow \operatorname{tg} x + \operatorname{tg} x \cos x - \operatorname{sen} x = 1 \rightarrow$$

$$\rightarrow \operatorname{tg} x + \frac{\operatorname{sen} x}{\cos x} \cos x - \operatorname{sen} x = 1 \rightarrow$$

$$\rightarrow \operatorname{tg} x = 1 \begin{cases} x_1 = 45^\circ + 360^\circ k \\ x_2 = 225^\circ + 360^\circ k \end{cases} \text{ con } k \in \mathbb{Z}$$

8. Simplifica:

$$\text{a) } \frac{\operatorname{sen} 60^\circ + \operatorname{sen} 30^\circ}{\cos 60^\circ + \cos 30^\circ}$$

$$\text{b) } \frac{\operatorname{sen}^2 \alpha}{1 - \cos \alpha} \left(1 + \operatorname{tg}^2 \frac{\alpha}{2}\right)$$

$$\text{a) } \frac{\operatorname{sen} 60^\circ + \operatorname{sen} 30^\circ}{\cos 60^\circ + \cos 30^\circ} = \frac{2\operatorname{sen} \frac{60^\circ + 30^\circ}{2} \cos \frac{60^\circ - 30^\circ}{2}}{2\cos \frac{60^\circ + 30^\circ}{2} \cos \frac{60^\circ - 30^\circ}{2}} = \frac{\operatorname{sen} 45^\circ}{\cos 45^\circ} = \operatorname{tg} 45^\circ = 1$$

$$\begin{aligned} \text{b) } \frac{\operatorname{sen}^2 \alpha}{1 - \cos \alpha} \left(1 + \operatorname{tg}^2 \frac{\alpha}{2}\right) &= \frac{\operatorname{sen}^2 \alpha}{1 - \cos \alpha} \left(1 + \frac{1 - \cos \alpha}{1 + \cos \alpha}\right) = \frac{\operatorname{sen}^2 \alpha}{1 - \cos \alpha} \left(\frac{2}{1 + \cos \alpha}\right) = \\ &= \frac{2\operatorname{sen}^2 \alpha}{1 - \cos^2 \alpha} = \frac{2\operatorname{sen}^2 \alpha}{\operatorname{sen}^2 \alpha} = 2 \end{aligned}$$